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The algebraic and analytic compactifications of the Hitchin moduli space

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Abstract

Following the work of Mazzeo–Swoboda–Weiß–Witt [Duke Math. J. 165 (2016), 2227–2271] and Mochizuki [J. Topol. 9 (2016), 1021–1073], there is a map $\overline{\Xi}$ between the algebraic compactification of the Dolbeault moduli space of $\mathrm{SL}(2,\mathbb{C})$ Higgs bundles on a smooth projective curve coming from the $\mathbb{C}^*$ action and the analytic compactification of Hitchin's moduli space of solutions to the $\mathsf{SU}(2)$ self-duality equations on a Riemann surface obtained by adding solutions to the decoupled equations, known as 'limiting configurations'. This map extends the classical Kobayashi–Hitchin correspondence. The main result that this article will show is that $\overline{\Xi}$ fails to be continuous at the boundary over a certain subset of the discriminant locus of the Hitchin fibration.

Contents

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1. Introduction

Let Σ be a closed Riemann surface of genus $g \geq 2$. The coarse Dolbeault moduli space of $\mathrm{SL}(2,\mathbb{C})$ semistable Higgs bundles on Σ, denoted by $\mathcal{M}_{\mathrm{Dol}}$, and Hitchin's moduli space of solutions to the $\mathsf{SU}(2)$ self-duality equations on Σ, denoted by $\mathcal{M}_{\mathrm{Hit}}$, have been extensively studied since their introduction more than 35 years ago. The Kobayashi–Hitchin correspondence, proved in [Hit87a], gives a homeomorphism between these two moduli spaces:

$$\Xi : \mathcal{M}_{\mathrm{Dol}} \xrightarrow{\sim} \mathcal{M}_{\mathrm{Hit}}. \tag{1}$$

Both spaces are noncompact: $\mathcal{M}_{\mathrm{Dol}}$ is naturally a quasiprojective variety [Nit91, Sim94], and like monopole moduli spaces, $\mathcal{M}_{\mathrm{Hit}}$ admits Higgs fields of arbitrarily large norms. Nevertheless, the map Ξ is proper. Recently, there has been interest from several directions on natural compactifications of these two spaces. A key feature on the Dolbeault side is the existence of a $\mathbb{C}^*$ action with the Białynicki–Birula property, and this may be used to define a completion of $\mathcal{M}_{\mathrm{Dol}}$ as a projective variety [Hau98, dC21, Fan22a]. The ideal points are identified with the $\mathbb{C}^*$ orbits in the complement of the nilpotent cone of $\mathcal{M}_{\mathrm{Dol}}$. The Hitchin moduli space also admits a more recently introduced compactification, $\overline{\mathcal{M}}_{\mathrm{Hit}}$, based on the work of several authors (see

[MSWW16, Moc16, Tau13b]). The boundary of $\overline{\mathcal{M}}_{\rm Hit}$ is given by gauge equivalence classes of limiting configurations. This compactification is relevant to many aspects of Hitchin's moduli space. For more details, we refer the reader to [DN19, MSWW14, Fre20, FMSW22, OSWW20, KNPS15, CL22] and to the references therein.

By the work of [MSWW16, Moc16], there is a natural extension

$$\overline{\Xi}:\overline{\mathcal{M}}_{\rm Dol}\longrightarrow\overline{\mathcal{M}}_{\rm Hit} \tag{2}$$

of the Kobayashi–Hitchin correspondence to the two compactifications described above, and it is of interest to study the geometry of this map. Doing so involves another key feature of Hitchin's moduli space; namely, spectral curves. Spectral curves and spectral data [Hit92] realise the Dolbeault moduli space as an algebraically complete integrable system $\mathcal{H}:\mathcal{M}_{\rm Dol}\to\mathcal{B}$. In the case of $\mathrm{SL}(2,\mathbb{C})$, the base $\mathcal{B}$ is the space of holomorphic quadratic differentials on Σ. Given $q\in H^0(K^2)$, one obtains a (scheme theoretic) spectral curve S_q. This curve is reduced if $q\neq 0$, is irreducible if q is not the square of an abelian differential, and is smooth if q has simple zeros. Let $\mathcal{B}^{\rm reg}\subset\mathcal{B}$ denote the open cone of quadratic differentials with simple zeros.

The ideal points of both compactifications $\overline{\mathcal{M}}_{\rm Dol}$ and $\overline{\mathcal{M}}_{\rm Hit}$ have associated nonzero quadratic differentials and, therefore, spectral curves. We write $\overline{\mathcal{M}}_{\rm Dol}^{\rm reg}$ for the elements in $\overline{\mathcal{M}}_{\rm Dol}$ with smooth spectral curves and $\overline{\mathcal{M}}_{\rm Dol}^{\rm sing}=\overline{\mathcal{M}}_{\rm Dol}\setminus\overline{\mathcal{M}}_{\rm Dol}^{\rm reg}$ for those with singular spectral curves; similarly for $\overline{\mathcal{M}}_{\rm Hit}^{\rm reg}$ and $\overline{\mathcal{M}}_{\rm Hit}^{\rm sing}$. We then have the following result:

Theorem 1.1. *The restriction of the compactified Kobayashi–Hitchin map $\overline{\Xi}$ to the locus with smooth associated spectral curves defines a homeomorphism $\overline{\mathcal{M}}_{\rm Dol}^{\rm reg}\simeq\overline{\mathcal{M}}_{\rm Hit}^{\rm reg}$. On the singular spectral curve locus, however, $\overline{\Xi}^{\rm sing}:\overline{\mathcal{M}}_{\rm Dol}^{\rm sing}\to\overline{\mathcal{M}}_{\rm Hit}^{\rm sing}$ is neither surjective nor injective.*

It will be convenient to analyse the behavior along rays in $\mathcal{B}$, where the spectral curve is simply rescaled. For $q\neq 0$ (a quadratic differential), we set $\overline{\mathcal{M}}_{{\rm Dol},q^+}$ (resp. $\overline{\mathcal{M}}_{{\rm Hit},q^+}$) to be the points in $\overline{\mathcal{M}}_{\rm Dol}$ (resp. $\overline{\mathcal{M}}_{\rm Hit}$) with spectral curves S_{tq}, $t\in\mathbb{R}^+$. The restriction of $\overline{\Xi}$ gives a map $\overline{\Xi}_{q^+}:\overline{\mathcal{M}}_{{\rm Dol},q^+}\to\overline{\mathcal{M}}_{{\rm Hit},q^+}$. We shall study the continuous behavior of $\overline{\Xi}_{q^+}$ for points in the fibre $\mathcal{H}^{-1}(tq)$ as $t\to+\infty$. For convenience, we set $\mathcal{M}_{q^+}:=\overline{\mathcal{M}}_{{\rm Dol},q^+}\cap\mathcal{M}_{\rm Dol}$. When q is irreducible, (that is, not a square), all elements in $\mathcal{M}_{q^+}$ are stable. Via the Hitchin [Hit87b] and Beauville–Narasimhan–Ramanan (BNR) correspondence [BNR89], this reduces the description of the fibre $\mathcal{M}_q:=\mathcal{H}^{-1}(q)$ to the characterisation of rank 1 torsion-free sheaves on the integral curve S_q.

In [Reg80], parameter spaces for rank 1 torsion-free sheaves on algebraic curves with Gorenstein singularities were studied in the context of compactified Jacobians, and the crucial notion of a parabolic module was introduced. This was extensively investigated by Cook in [Coo93, Coo98], partially following ideas of Bhosle [Bho92]. For simple plane curve singularities of the type appearing in spectral curves, one makes use of the local classification of torsion-free modules of Greuel–Knörrer [GK85]. These methods were applied to study the Hitchin fibration by Gothen–Oliveira in [GO13] (see also [KSZ22] for a recent study). In parallel, Horn [Hor22a] defines a stratification $\mathcal{M}_q=\bigcup_D\mathcal{M}_{q,D}$ labelled by certain effective divisors contained in the divisor of q called *σ-divisors* (see Section 5.5, and also [HN] for the more general situation).

Using the results from these references, we reinterpret the work of Mochizuki [Moc16] and Mochizuki–Szabó [MS23]. We first prove that the restriction of the compactified Kobayashi–Hitchin map to the boundary is discontinuous in general. Following that, by utilising the exponential decay results from Mochizuki–Szabó [MS23], which play an essential role, we demonstrate that the entirety of $\overline{\Xi}_{q^+}$ is discontinuous.

THEOREM 1.2. *Let $q \neq 0$ be an irreducible quadratic differential.*

(i) *The boundary map $\partial\overline{\Xi}_{q^+}|_{\partial\overline{\mathcal{M}}^{\mathrm{st}}_{\mathrm{Dol},q^+}}$ is continuous if q has zeros only of odd order and is discontinuous if q has at least one zero of even order.*

(ii) *If q has at least one zero of even order, then for each σ-divisor $D \neq 0$, there exists an even integer $n_D \geq 1$ so that for any Higgs bundle $(\mathcal{F}, \psi) \in \mathcal{M}_{q,D}$, there exist $2n_D$ sequences of Higgs bundles $(\mathcal{E}_i^k, \varphi_i^k)$, $k = 1, \ldots, 2n_D$ such that*

- $\lim_{i\to\infty}(\mathcal{E}_i^k, \varphi_i^k) = (\mathcal{F}, \psi)$ *for* $k = 1, \ldots, 2n_D$,
- *and if we write*

$$\eta^k := \lim_{i\to\infty} \partial\overline{\Xi}_{q^+}(\mathcal{E}_i^k, t_i\varphi_i^k) \quad , \quad \xi := \lim_{i\to\infty} \partial\overline{\Xi}_{q^+}(\mathcal{F}, t_i\psi)$$

  * *and if $(\mathcal{F}, \psi)$ doesn't lie in the real locus, then $\xi, \eta^1, \ldots, \eta^{2n_D}$ are $2n_D + 1$ different limiting configurations;*
  * *if $(\mathcal{F}, \psi)$ lies in the real locus, then $\eta^i \cong \eta^{n_D+i}$ for $i = 1, \cdots, n$, and we obtain $n_D + 1$ different limiting configurations.*
- *for each k, there exist constants $t_i \to +\infty$ such that $\lim_{i\to\infty} \overline{\Xi}_{q^+}(\mathcal{E}_i^k, t_i\varphi_i^k) \neq \overline{\Xi}_{q^+}(\mathcal{F}, \psi)$.*

When q is reducible, the description of Higgs bundles in the fibre over q becomes more complicated because of, among other things, the existence of strictly semistable objects. To understand this, we use the local descriptions of Gothen–Oliveira and Mochizuki (see [GO13, Moc16]). In contrast to the irreducible case, the analogous exponential decay result to that of Mochizuki–Szabó [Moc16] is, unfortunately, currently not available. This results in a weaker statement for the reducible fibre. Recall that we have defined $\overline{\Xi}_{q^+} : \overline{\mathcal{M}}_{\mathrm{Dol},q^+} \to \overline{\mathcal{M}}_{\mathrm{Hit},q^+}$ as the compactified Kobayashi–Hitchin map and $\partial\overline{\Xi}_{q^+} : \partial\overline{\mathcal{M}}_{\mathrm{Dol},q^+} \to \partial\overline{\mathcal{M}}_{\mathrm{Hit},q^+}$ as its restriction to the compactified boundary. With this notation, the following holds:

THEOREM 1.3. *Suppose that $q \neq 0$ is reducible; if $g \geq 3$, then the boundary map $\partial\overline{\Xi}_{q^+}|_{\partial\overline{\mathcal{M}}^{\mathrm{st}}_{\mathrm{Dol},q^+}}$ is discontinuous. However, if $g = 2$, the boundary map $\partial\overline{\Xi}_{q^+}|_{\partial\overline{\mathcal{M}}^{\mathrm{st}}_{\mathrm{Dol},q^+}}$ is continuous.*

This article is organized as follows: In Section 2, we provide a brief overview of Higgs bundles and the BNR correspondence. In Section 3, we introduce the concepts of filtered bundles and their compactness properties. Section 4 defines the algebraic and analytic compactifications. Section 5 introduces parabolic modules and examines their connection to spectral curves. The main results for Hitchin fibres with irreducible singular spectral curves are established in Section 6. In Section 7, the results for the reducible case are proven. Finally, in Section 8, we construct the compactified Kobayashi–Hitchin map and prove the main results. The Appendix, based on the work of Greuel–Knörrer, calculates some invariants of rank 1 torsion-free sheaves on the spectral curves we consider.

2. Background on Higgs bundles

This section gives a very brief overview of the Dolbeault and Hitchin moduli spaces, spectral curve descriptions and the nonabelian Hodge correspondence. For more details on these topics, see [Hit87a, Hit87b, Sim92].

2.1 Higgs bundles

As in the Introduction, throughout this paper Σ will denote a closed Riemann surface of genus $g \geq 2$, with structure sheaf $\mathcal{O} = \mathcal{O}_\Sigma$ and canonical bundle $K = K_\Sigma$. Let $E \to \Sigma$ be a complex vector bundle. A Higgs bundle consists of a pair $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is a holomorphic bundle structure on E and where $\varphi \in H^0(\mathrm{End}(\mathcal{E}) \otimes K)$. If $\mathrm{rank}(E) = 1$, then a Higgs field is just an abelian differential ω. The pair $(\mathcal{E}, \varphi)$ is called an $\mathrm{SL}(2,\mathbb{C})$ Higgs bundle if $\mathrm{rank}(E) = 2$, $\det(\mathcal{E})$ has a fixed isomorphism with the trivial bundle and if $\mathrm{Tr}(\varphi) = 0$. In this article we will focus mainly on $\mathrm{SL}(2,\mathbb{C})$ Higgs bundles, but the rank 1 case will also be important.

Let $(\mathcal{E}, \varphi)$ be an $\mathrm{SL}(2,\mathbb{C})$ Higgs bundle. A (proper) Higgs subbundle of $(\mathcal{E}, \varphi)$ is a holomorphic line bundle $\mathcal{L} \subset \mathcal{E}$ that is φ-invariant; that is, $\varphi : \mathcal{L} \to \mathcal{L} \otimes K$. In this case the restriction $\varphi_{\mathcal{L}} := \varphi|_{\mathcal{L}}$ makes $(\mathcal{L}, \varphi_{\mathcal{L}})$ a rank 1 Higgs bundle. Moreover, φ induces a Higgs bundle structure on the quotient $\mathcal{E}/\mathcal{L}$. We say that $(\mathcal{E}, \varphi)$ is stable (resp. semistable) if for all Higgs subbundles $\mathcal{L}$, $\deg \mathcal{L} < 0$ (resp. $\deg \mathcal{L} \leq 0$). We say that $(\mathcal{E}, \varphi)$ is polystable if $(\mathcal{E}, \varphi) \simeq (\mathcal{L}, \omega) \oplus (\mathcal{L}^{-1}, -\omega)$, where $\mathcal{L}$ is a degree-zero holomorphic line bundle and $\omega \in H^0(K)$.

If $(\mathcal{E}, \varphi)$ is strictly semistable (that is, semistable but not polystable), the Seshadri filtration [Ses67] gives the unique Higgs subbundle $0 \subset (\mathcal{L}, \omega) \subset (\mathcal{E}, \varphi)$, with $\deg(\mathcal{L}) = \frac{1}{2}\deg(\mathcal{E}) = 0$. If we write $(\mathcal{L}', \omega') := (\mathcal{E}, \varphi)/(\mathcal{L}, \omega)$, then we have $\omega' = -\omega$ and $\mathcal{L}' = \mathcal{L}^{-1}$. The associated graded bundle $\mathrm{Gr}(\mathcal{E}, \varphi) = (\mathcal{L}, \omega) \oplus (\mathcal{L}^{-1}, -\omega)$ of this filtration is a polystable $\mathrm{SL}(2,\mathbb{C})$ Higgs bundle. We say that $(\mathcal{E}, \varphi)$ is S-equivalent to $\mathrm{Gr}(\mathcal{E}, \varphi)$.

Holomorphic bundles $\mathcal{E}$, with underlying C^∞ bundle E, are in 1-to-1 correspondence, with $\bar{\partial}$-operators $\bar{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$. We use the notation $\mathcal{E} := (E, \bar{\partial}_E)$. Let $\mathcal{C}$ denote the space of pairs $(\bar{\partial}_E, \varphi)$, $\bar{\partial}_E \varphi = 0$. Let $\mathcal{C}^s$ and $\mathcal{C}^{ss}$ denote the subspaces of $\mathcal{C}$ where the Higgs bundles are stable (resp. semistable). The complex gauge transformation group $\mathcal{G}_\mathbb{C} := \mathrm{Aut}(E)$ has a right-hand action on $\mathcal{C}$ by defining for $g \in \mathcal{G}_\mathbb{C}$, $(\bar{\partial}_E, \varphi)g := (g^{-1} \circ \bar{\partial} \circ g, g^{-1} \circ \varphi \circ g)$.

There is a quasiprojective scheme $\mathcal{M}_{\mathrm{Dol}}$ whose closed points are in 1-to-1 correspondence with isomorphism classes of polystable $\mathrm{SL}(2,\mathbb{C})$ Higgs bundles constructed via (finite dimensional) Geometric Invariant Theory (see [Nit91, Sim94]). In [Fan22b] it was shown that the infinite dimensional quotient $\mathcal{C}^{\mathrm{ss}}//\mathcal{G}_\mathbb{C}$, where the double slash indicates that S-equivalent orbits are identified, admits the structure of a complex analytic space that is biholomorphic to the analytification $\mathcal{M}_{\mathrm{Dol}}^{\mathrm{an}}$ of $\mathcal{M}_{\mathrm{Dol}}$. Henceforth, we shall work in the complex analytic category; identify the algebro–geometric and gauge theoretic moduli spaces as complex analytic spaces; and simply denote them both by $\mathcal{M}_{\mathrm{Dol}}$. We note that the set of stable Higgs bundles modulo gauge transformations, $\mathcal{M}_{\mathrm{Dol}}^s := \mathcal{C}^{\mathrm{s}}/\mathcal{G}_\mathbb{C}$, is a geometric quotient and an open subset of $\mathcal{M}_{\mathrm{Dol}}$.

Finally, notice that the pair $(\mathcal{E}, \varphi)$ is stable (resp. semistable) if and only if the same is true for $(\mathcal{E}, \lambda\varphi)$, $\lambda \in \mathbb{C}^*$. Hence, $\mathcal{M}_{\mathrm{Dol}}$ admits an action of $\mathbb{C}^*$ that preserves $\mathcal{M}_{\mathrm{Dol}}^s$. Although $\mathcal{M}_{\mathrm{Dol}}$ is only quasiprojective, the $\mathbb{C}^*$ action satisfies the Białynicki–Birula property:

Theorem 2.1. *For any* $[(\mathcal{E}, \varphi)] \in \mathcal{M}_{\mathrm{Dol}}$,

$$\lim_{\lambda \to 0} \lambda \cdot [(\mathcal{E}, \varphi)] := \lim_{\lambda \to 0} [(\mathcal{E}, \lambda\varphi)]$$

exists in $\mathcal{M}_{\mathrm{Dol}}$.

2.2 Spectral curves and the Hitchin fibration

The Hitchin map is defined as

$$\mathcal{H} : \mathcal{M}_{\mathrm{Dol}} \longrightarrow H^0(K^2) \quad [(\mathcal{E}, \varphi)] \mapsto \det(\varphi)$$

where $H^0(K^2) =: \mathcal{B}$ is known as the Hitchin base. Hitchin [Hit87a, Hit87b] showed that $\mathcal{H}$ is a proper map and a fibration by abelian varieties over the open cone $\mathcal{B}^{\text{reg}} \subset \mathcal{B}$ consisting of nonzero quadratic differentials with only simple zeros. The discriminant locus $\mathcal{B}^{\text{sing}} := \mathcal{B} \setminus \mathcal{B}^{\text{reg}}$ consists of quadratic differentials that either are identically zero or have at least one zero with multiplicity. For $q \in \mathcal{B}$, let $\mathcal{M}_q := \mathcal{H}^{-1}(q)$. The 'most singular fibre' $\mathcal{M}_0$ is called the *nilpotent cone*.

Consider the total space $\text{Tot}(K)$ of K, along with its projection $\pi : \text{Tot}(K) \to \Sigma$. The pullback bundle $\pi^* K$ has a tautological section, which we denote by $\lambda \in H^0(\text{Tot}(K), \pi^* K)$. Given any $q \neq 0 \in H^0(K^2)$, the *spectral curve* S_q associated with q is the zero scheme of the section $\lambda^2 - \pi^* q \in H^0(\text{Tot}(K), \pi^* K)$. This is a reduced, but possibly reducible, projective algebraic curve. The restriction of π to S_q, also denoted by $\pi : S_q \to \Sigma$, is a double covering branched along the zeros of q.

The spectral curve S_q is smooth if and only if q has only simple zeros. It is reducible if and only if $q = -\omega \otimes \omega$ for some $\omega \in H^0(K)$. In the latter case, we call such quadratic differentials *reducible*, and we otherwise refer to them as *irreducible*. There is a noteworthy observation regarding irreducible spectral curves.

Proposition 2.2. *Let $(\mathcal{E}, \varphi)$ be a Higgs bundle with $q = \det(\varphi)$, and suppose that q is irreducible. Then $(\mathcal{E}, \varphi)$ has no proper invariant subbundles. In particular, $(\mathcal{E}, \varphi)$ is stable.*

Proof. Suppose $\mathcal{L} \subset \mathcal{E}$ is φ-invariant, and let $\varphi_{\mathcal{L}}$ be the restriction. Then

$$\det \varphi = -\frac{1}{2}\text{Tr}(\varphi^2) = -(\varphi_{\mathcal{L}})^2$$

which contradicts the assumption. □

Let us emphasise that being reducible is not the same as having only even zeros. To see this, suppose that $\text{Div}(q) = 2D$. Then $K \simeq \mathcal{O}(D) \otimes \mathcal{I}$, where $\mathcal{I}$ is a 2-torsion point in the Jacobian. The spectral curve S_q is reducible if and only if $\mathcal{I}$ is trivial.

2.3 Rank 1 torsion-free sheaves and the BNR correspondence

In this subsection, we provide some background on rank 1 torsion-free sheaf theory over spectral curves in the context of the Hitchin and BNR correspondence, as developed in [Hit87b, BNR89].

Let S be a reduced and irreducible complex projective curve and $\mathcal{O}_S$ its structure sheaf. The moduli space of invertible sheaves on S is denoted by $\text{Pic}(S)$. If $\mathcal{F}$ is a coherent analytic sheaf on S, we can define its cohomology groups $H^i(S, \mathcal{F})$. Since $\dim S = 1$, $H^i(S, \mathcal{F}) = 0$ for $i \geq 2$. The Euler characteristic is defined as $\chi(\mathcal{F}) = \dim H^0(S, \mathcal{F}) - \dim H^1(S, \mathcal{F})$. The degree of a torsion-free sheaf $\mathcal{F}$ is given by $\deg(\mathcal{F}) = \chi(\mathcal{F}) - \text{rank}(\mathcal{F})\chi(\mathcal{O}_S)$. If $\mathcal{F}$ is locally free, then $\deg(\mathcal{F})$ coincides with the degree of the invertible sheaf $\det(\mathcal{F})$. We let $\text{Pic}^d(S) \subset \text{Pic}(S)$ denote the degree d component.

Let $\overline{\text{Pic}}^d(S)$ be the moduli space of degree d, rank 1 torsion-free sheaves on S, and let $\overline{\text{Pic}}(S) = \prod_{d \in \mathbb{Z}} \overline{\text{Pic}}^d(S)$ [D'S79]. Then, $\overline{\text{Pic}}^d(S)$ is an irreducible projective scheme containing $\text{Pic}^d(S)$ as an open subscheme. When S is smooth, we have $\overline{\text{Pic}}^d(S) = \text{Pic}^d(S)$. The relationship to Higgs bundles is given by the following:

Theorem 2.3. *Let $q \in H^0(K^2)$ be an irreducible quadratic differential with spectral curve S_q. There is then a bijective correspondence between points in $\overline{\text{Pic}}(S_q)$ and isomorphism classes of rank 2 Higgs pairs $(\mathcal{E}, \varphi)$ with $\text{Tr}(\varphi) = 0$ and $\det(\varphi) = q$. Explicitly: If $\mathcal{L} \in \overline{\text{Pic}}(S_q)$, then*

$\mathcal{E} := \pi_*(\mathcal{L})$ is a rank 2 vector bundle, and the homomorphism $\pi_*\mathcal{L} \to \pi_*\mathcal{L} \otimes K \cong \pi_*(\mathcal{L} \otimes \pi^*K)$, given by multiplication by the canonical section λ, defines the Higgs field φ.

This correspondence gives the very useful exact sequence

$$0 \to \mathcal{L} \otimes \mathcal{I} \to \pi^*\mathcal{E} \xrightarrow{\pi^*\varphi-\lambda} \pi^*\mathcal{E} \otimes \pi^*K \to \mathcal{L} \otimes \pi^*K \to 0 \tag{3}$$

for some ideal sheaf $\mathcal{I}$. If S is smooth, then $\mathcal{I} = \mathcal{O}_S(-\Delta)$, where Δ is the ramification divisor. The sequence (3) will be used in Section 6.

Let q be a quadratic differential with only simple zeros, and to simplify notation, write $S = S_q$. Let $\Lambda := \mathrm{Div}(\lambda)$ be the ramification divisor of the map $\pi : S \to \Sigma$. By the Riemann–Hurwitz formula, the genus of S is $g(S) = 4g-3$, where g is the genus of Σ. Furthermore, for any $\mathcal{L} \in \mathrm{Pic}(S)$, Riemann–Roch gives $\deg(\pi_*\mathcal{L}) = \deg(\mathcal{L}) - (2g-2)$. The $\mathrm{SL}(2,\mathbb{C})$ Higgs bundles are characterized by

$$\mathcal{T} := \{\mathcal{L} \in \mathrm{Pic}^{2g-2}(S) \mid \det(\pi_*\mathcal{L}) = \mathcal{O}_\Sigma\}. \tag{4}$$

By the Hitchin–BNR correspondence (Theorem 2.3), the map $\chi_{\mathrm{BNR}} : \mathcal{T} \to \mathcal{M}_q$ is a bijection.

The branched double cover $\pi : S \to \Sigma$ is given by an involution $\sigma : S \to S$. We have the norm map $\mathrm{Nm}_{S/\Sigma} : \mathrm{Jac}(S) \to \mathrm{Jac}(\Sigma)$, where $\mathrm{Jac}(S)$ is the connected component of the trivial line bundle in $\mathrm{Pic}(S)$ and $\mathrm{Nm}_{S/\Sigma}(\mathcal{O}_S(D)) := \mathcal{O}_\Sigma(\pi(D))$. The Prym variety is defined as

$$\mathrm{Prym}(S/\Sigma) := \ker(\mathrm{Nm}_{S/\Sigma}) = \{\mathcal{L} \in \mathrm{Pic}(S) \mid \mathcal{L} \otimes \sigma^*\mathcal{L} = \mathcal{O}_S\}.$$

Also, we have $\det(\pi_*\mathcal{L}) \cong \mathrm{Nm}_{S/\Sigma}(\mathcal{L}) \otimes K^{-1}$. Thus, $\mathcal{T}$ can be expressed as

$$\mathcal{T} = \{\mathcal{L} \in \mathrm{Pic}^{2g-2}(S) \mid \mathrm{Nm}_{S/\Sigma}(\mathcal{L}) \cong K\}.$$

Hence, $\mathcal{T}$ is a torsor over $\mathrm{Prym}(S/\Sigma)$. Explicitly, by choosing $\mathcal{L}_0 \in \mathcal{T}$, we obtain an isomorphism $\mathcal{T} \xrightarrow{\sim} \mathrm{Prym}(S/\Sigma)$ given by $\mathcal{L} \to \mathcal{L} \otimes \mathcal{L}_0^{-1}$.

To summarize, we have the following:

Proposition 2.4. *Let q be a quadratic differential with simple zeros. Then $\mathcal{M}_q \cong \mathcal{T} \cong \mathrm{Prym}(S/\Sigma)$.*

If $q \neq 0$ is irreducible but nongeneric, the spectral curve S is singular and irreducible. We may still define the set $\overline{\mathcal{T}} \subset \overline{\mathrm{Pic}}^{2g-2}(S)$ as follows:

$$\overline{\mathcal{T}} := \left\{\mathcal{L} \in \overline{\mathrm{Pic}}^{2g-2}(S) \mid \det(\pi_*\mathcal{L}) \cong \mathcal{O}_\Sigma\right\}\ .$$

We also set $\mathcal{T} := \overline{\mathcal{T}} \cap \mathrm{Pic}^{2g-2}$. Then $\overline{\mathcal{T}}$ is the natural compactification of $\mathcal{T}$ induced by the inclusion $\mathrm{Pic}^{2g-2}(S) \subset \overline{\mathrm{Pic}}(S)$. The BNR correspondence, as stated in Theorem 2.3, implies that $\chi_{\mathrm{BNR}} : \overline{\mathcal{T}} \to \mathcal{M}_q$ is an isomorphism.

2.4 The Hitchin moduli space and the nonabelian Hodge correspondence

We now recall the well-known nonabelian Hodge correspondence (NAH), which relates the space of flat $\mathrm{SL}(2,\mathbb{C})$ connections, Higgs bundles and solutions to the Hitchin equations. This result was developed in the work of Hitchin [Hit87a], Simpson [Sim88], Corlette [Cor88] and Donaldson [Don87].

As above, let E be a trivial(ised), smooth, rank 2 vector bundle over the Riemann surface Σ, and let H_0 be a fixed Hermitian metric on E. We denote by $\mathfrak{sl}(E)$ (resp. $\mathfrak{su}(E)$) the bundle of traceless (resp. traceless skew-Hermitian) endomorphisms of E. Let A be a unitary (with respect to H_0) connection on E that induces the trivial connection on $\det E$, and let $\phi \in \Omega^1(i\mathfrak{su}(E))$. We will sometimes also refer to ϕ as a Higgs field. The Hitchin equations for the pair (A, ϕ) are given by

$$F_A + \phi \wedge \phi = 0 \quad d_A\phi = d_A^*\phi = 0. \tag{5}$$

If we split the Higgs field into type $\phi = \varphi + \varphi^\dagger$, with $\varphi \in \Omega^{1,0}(\mathfrak{sl}(E))$, then (5) is equivalent to

$$F_A + [\varphi, \varphi^\dagger] = 0 \quad \bar{\partial}_A\varphi = 0. \tag{6}$$

Notice that $(\bar{\partial}_E, \varphi)$ then defines an $\mathrm{SL}(2,\mathbb{C})$ Higgs bundle. The Hitchin moduli space, denoted by $\mathcal{M}_{\mathrm{Hit}}$, is the moduli space of solutions to the Hitchin equations, which is given by

$$\mathcal{M}_{\mathrm{Hit}} := \{(A, \phi) \mid (A, \phi) \text{ satisfies } (5)\}/\mathcal{G},$$

where $\mathcal{G}$ is the gauge group of unitary automorphisms of E. Recall that a flat connection $\mathcal{D}$ is called completely reducible if and only if it is a direct sum of irreducible flat connections. The NAH can be summarized as follows:

Theorem 2.5. *A Higgs bundle $(\mathcal{E}, \varphi)$ is polystable if and only if there exists a Hermitian metric H such that the corresponding Chern connection A and Higgs field $\phi = \varphi + \varphi^\dagger$ solve the Hitchin equations (5). Moreover, the connection $\mathcal{D}$ defined by $\mathcal{D} = \nabla_A + \phi$ is a completely reducible flat connection, and it is irreducible if and only if $(\mathcal{E}, \varphi)$ is stable.*

Conversely, a flat connection $\mathcal{D}$ is completely reducible if and only if there exists a Hermitian metric H on E such that when we express $\mathcal{D} = \nabla_A + \varphi + \varphi^\dagger$, we have $\bar{\partial}_{\mathcal{E}}\varphi = 0$. Moreover, the corresponding Higgs bundle $(\mathcal{E}, \varphi)$ is polystable, and it is stable if and only if $\mathcal{D}$ is irreducible.

The nonabelian Hodge correspondence gives the Kobayashi–Hitchin homeomorphism (1), which, when restricted to the stable locus, is a diffeomorphism onto irreducible solutions of (5).

Finally, we note that there is an action of S^1 on $\mathcal{M}_{\mathrm{Hit}}$ defined by $(A, \phi) \to (A, e^{i\theta} \cdot \phi)$, where $e^{i\theta} \cdot \phi = e^{i\theta}\varphi + e^{-i\theta}\varphi^\dagger$. With respect to this and the $S^1 \subset \mathbb{C}^*$ action on $\mathcal{M}_{\mathrm{Dol}}$, the map Ξ is S^1-equivariant.

3. Filtered bundles and compactness

Filtered (or parabolic) bundles are described, for example, in [Sim90]. They play a key role in the analytic compactification. This section provides a brief overview of filtered line bundles and demonstrates a compactness result.

3.1 Filtered line bundles

Let Z be a finite collection of distinct points on a closed Riemann surface Σ, and let $\Sigma' = \Sigma \setminus Z$. Viewing Σ as a projective algebraic curve, an algebraic line bundle L over the affine curve Σ' is a line bundle defined by regular transition functions on Zariski open sets over Σ'. The sheaf of sections of L can be extended in infinitely many different ways over Z to obtain (invertible) coherent analytic sheaves on Σ. The sections of L are then realised as meromorphic sections of any such extensions that are regular on Σ'.

A *filtered line bundle* $\mathcal{F}_*(L)$ is an algebraic line bundle $L \to \Sigma'$, along with a collection $\{L_\alpha\}_{\alpha\in\mathbb{R}}$ of coherent extensions across the punctures Z, such that $L_\alpha \subset L_\beta$ for $\alpha \geq \beta$ for fixed, sufficiently small ϵ, $L_{\alpha-\epsilon} = L_\alpha$ and $L_\alpha = L_{\alpha+1} \otimes \mathcal{O}_\Sigma(Z)$. Let $\mathrm{Gr}_\alpha = L_\alpha / L_{\alpha+\epsilon}$ denote the quotient (torsion) sheaf. A value α where $\mathrm{Gr}_\alpha \neq 0$ is called a jump. Since we are considering line bundles, for each p in the support of Gr_{α_p}, there is exactly one jump α_p in the interval $[0,1)$. The collection of jumps α_p, $p \in Z$, fully determines the filtered bundle structure. If we denote by $\mathcal{L} := L_0$, the degree of a filtered line bundle is defined as

$$\deg(\mathcal{F}_*(L)) := \deg(\mathcal{L}) + \sum_{p \in Z} \alpha_p.$$

Alternatively, a *weighted line bundle* is a pair $(\mathcal{L}, \chi)$ for which $\mathcal{L} \to \Sigma$ is a holomorphic line bundle and $\chi : Z \to \mathbb{R}$ is a weight function. The degree of a weighted bundle is defined as

$$\deg(\mathcal{L}, \chi) := \deg(\mathcal{L}) + \sum_{p \in Z} \chi_p.$$

The notions of filtered and weighted line bundles are nearly equivalent: Namely, given a filtered line bundle $\mathcal{F}_*(L)$, we define $\mathcal{L} := L_0$ and $\chi_p = \alpha_p$. Conversely, given a weighted line bundle $(\mathcal{L}, \chi)$, we let $\alpha_p = \chi_p + n_p$, where $n_p \in \mathbb{Z}$ is the unique integer and $0 \le \chi_p + n_p < 1$. A filtered bundle $\mathcal{F}_*(L)$, $L := \mathcal{L}|_{\Sigma'}$, is then determined by setting $L_0 = \mathcal{L}(-\sum_{p \in Z} n_p p)$ with jumps α_p. Clearly, $\deg(\mathcal{F}(L)) = \deg(\mathcal{L}, \chi)$. We shall use the notation $\mathcal{F}_*(\mathcal{L}, \chi)$ for the filtered bundle associated to a weighted bundle $(\mathcal{L}, \chi)$ in this way.

Different weighted bundles can give rise to the same filtered bundle. The following is a fact that will be frequently used in this article. If $D = \sum_{x \in Z} d_x x$ is a divisor supported on Z, let

$$\chi_D(x) := \begin{cases} d_x & x \in Z \\ 0 & x \in \Sigma \setminus Z. \end{cases}$$

Then for any weighted bundle $(\mathcal{L}, \chi)$, we have $\mathcal{F}_*(\mathcal{L}(D), \chi - \chi_D) = \mathcal{F}_*(\mathcal{L}, \chi)$.

Let $(\mathcal{L}_1, \chi_1)$ and $(\mathcal{L}_2, \chi_2)$ be two weighted lines bundles. We define the tensor product

$$(\mathcal{L}_1, \chi_1) \otimes (\mathcal{L}_2, \chi_2) := (\mathcal{L}_1 \otimes \mathcal{L}_2, \chi_1 + \chi_2)\ .$$

Then the degree is additive on tensor products. For filtered bundles, we define

$$\mathcal{F}_*(\mathcal{L}_1, \chi_1) \otimes \mathcal{F}_*(\mathcal{L}_2, \chi_2) := \mathcal{F}_*(\mathcal{L}_1 \otimes \mathcal{L}_2, \chi_1 + \chi_2). \tag{7}$$

The degree is again additive for the tensor product of filtered bundles. This agrees with the usual definition of tensor product for parabolic bundles.

3.2 Harmonic metrics for weighted line bundles

Proposition 3.1. *Let $(\mathcal{L}, \chi)$ be a degree-0 weighted bundle. Then there exists a Hermitian metric h on $\mathcal{L}_{\Sigma'}$, which is unique up to a multiplication by a nonzero constant, such that:*

(i) the Chern connection A_h of $(\mathcal{L}, h)$ is flat: $F_{A_h} = 0$;

(ii) for $p \in Z$ and (U_p, z), a holomorphic coordinate centered at p, $|z|^{-2\chi_p} h$ extends to a $\mathcal{C}^\infty$ Hermitian metric on $\mathcal{L}|_{U_p}$.

Proof. We first choose a background Hermitian metric h_0 such that $|z|^{-2\chi_p} h_0$ defines a $\mathcal{C}^\infty$ Hermitian metric defined on U_p. Let A_{h_0} be the Chern connection, and let F_{A_0} be the curvature. Note that F_{A_0} is smooth on Σ. By the Poincaré–Lelong formula, we have $\frac{\sqrt{-1}}{2\pi} \int_\Sigma F_{A_0} = \deg(\mathcal{L}, \chi) = 0$. Therefore, there exists a $\mathcal{C}^\infty$ function ρ such that $\Delta \rho + \frac{\sqrt{-1}}{2\pi} \Lambda F_{A_0} = 0$. We define $h = h_0 e^\rho$. For the corresponding Chern connection A_h, we have $F_{A_h} = 0$, which implies (i). Then (ii) follows from the property for h_0 since ρ is a smooth function on Σ. As ρ is well-defined up to a constant, h is also well-defined up to a constant, which implies the uniqueness of h up to a constant. □

The metric obtained above is called the *harmonic metric*. For a weighted bundle $(\mathcal{L},\chi)$, the holomorphic bundle $\mathcal{L}$ and the harmonic metric h define a filtration as follows: Given α, define the sheaf

$$L_\alpha(U) := \{s \in H^0(U,\mathcal{L}(*Z)) \mid |s|_h = O(r^{\alpha-\epsilon}) \text{ for all } \epsilon > 0\}$$

and any open set $U \subset \Sigma$. Here, r denotes the distance to Z in any smooth, conformal metric on Σ. It is straightforward to check that this defines a filtered bundle that matches $\mathcal{F}_*(\mathcal{L},\chi)$ under the correspondence given in the previous section.

Even though the harmonic metric is well-defined only up to a constant, the Chern connection $A = (\mathcal{L}, h)$ is independent of this choice. The $(1,0)$ part of A, denoted ∇_h, then defines logarithmic the connections $\nabla_h : L_\alpha \to L_\alpha \otimes K(Z)$.

3.3 Convergence of weighted line bundles

In this subsection, we consider the convergence of weighted line bundles. The main result we prove here is a consequence of [MS23, Theorem 1.8]. For the reader's convenience, we present a short proof in our situation.

Let (Σ_0, g_0) be a metrised Riemann surface (that is, a Riemann surface Σ_0 with conformal metric g_0). We view Σ_0 as given by an underlying surface C with almost complex structure J_0. Consider a neighbourhood U_1 of J_0 in the moduli space of holomorphic structures and a neighbourhood U_2 of g_0 in the space of smooth metrics. We denote the product of these neighbourhoods by $U = U_1 \times U_2$. We can define the fibre bundle $\mathrm{Pic}_U \to U$, where each fibre is the Picard group defined by the holomorphic structure. Let $(\Sigma_t = (C, J_t), g_t)$ be a family of metrised Riemann surfaces that converge smoothly to (Σ_0, g_0) as $t \to 0$. Let $Z_t \subset \Sigma_t$ be a collection of a finite number of points that converge to Z_0 in suitable symmetric products of C. For each $p \in Z_0$, we can write $Z_t = \cup_{p \in Z_0} Z_{t,p}$ such that all points in $Z_{t,p}$ converge to p. We define the convergence of weighted line bundles as follows:

DEFINITION 3.2. *A family of weighted line bundles $(\mathcal{L}_t, \chi_t)$ over $\Sigma_t \setminus Z_t$, with weights $\chi_t : Z_t \to \mathbb{R}$, converges to $(\mathcal{L}_0, \chi_0)$ if*

(i) *$\mathcal{L}_t$ converges to $\mathcal{L}_0$ in* Pic_U and if

(ii) *for all $p \in Z_0$ and t sufficiently small, $\sum_{q \in Z_{t,p}} \chi_t(q) = \chi_0(p)$.*

A sequence of filtered bundles $\mathcal{F}_*(\mathcal{L}_t)$ converges to $\mathcal{F}_*(\mathcal{L}_0)$ if the corresponding weighted bundles converge. The following theorem provides insight into the compactness of a sequence of weighted line bundles:

THEOREM 3.3. *Consider a family of weighted line bundles $(\mathcal{L}_t, \chi_t)$ defined over $(\Sigma_t \setminus Z_t)$ and with $\deg(\mathcal{L}_t, \chi_t) = 0$. Let h_t be the corresponding harmonic metrics from Proposition 3.1. If Z_t converges to Z_0, we write $Z_t = \cup_{p \in Z_0} Z_{t,p}$. Then there exists a weighted line bundle $(\mathcal{L}_0, \chi_0)$ over Z_0 with a harmonic metric h_0 such that*

(i) *After rescaling by $c_t > 0$, $c_t h_t$ converges to h_0 over $\Sigma_0 \setminus Z_0$ in the $\mathcal{C}^\infty_{\mathrm{loc}}$ sense.*

(ii) *If A_{h_t} is the Chern connection of $(\mathcal{L}_t, h_t)$, then on $\Sigma_0 \setminus Z_0$, $\lim_{t\to 0} \nabla_t = \nabla_0$ in $\mathcal{C}^\infty_{\mathrm{loc}}$.*

Proof. By the assumptions on weights, $\deg(\mathcal{L}_t)$ is a fixed, t-independent constant. Let $\gamma_t = (J_t, g_t)$ be a path in U. Then $\mathrm{Pic}_U|_{\gamma_t}$ is compact, and there exists an $\mathcal{L}_0 \in \mathrm{Pic}(\Sigma_0)$ such that $\mathcal{L}_t$ converges to $\mathcal{L}_0$. For $p \in Z_0$, define $\chi_0(p) = \sum_{q \in Z_{t,p}} \chi_t(q)$, after which you will obtain a weighted line bundle $(\mathcal{L}_0, \chi_0)$. Choose a family of approximate harmonic metrics h_t^{app} such that $|z|^{-2\chi_p} h_t^{\mathrm{app}}$ extends to a smooth metric in a neighbourhood of p, and h_t^{app} converges to h_0^{app} in $\mathcal{C}^\infty_{\mathrm{loc}}(\Sigma_0 \setminus Z_0)$. Moreover, write $h_t = h_t^{\mathrm{app}} e^{s_t}$. After a suitable rescaling of h_t, we can assume $\|s_t\|_{L^2} = 1$. Let $\rho_t := \Delta_t h_t^{\mathrm{app}}$ be the curvature defined by the metric h_t^{app}. Then s_t satisfies the equation $\Delta_t s_t = \rho_t$ over Σ. As ρ_t converges to $\rho_0 \in \mathcal{C}^\infty_{\mathrm{loc}}(\Sigma \setminus Z_0)$ and as g_t is a family with bounded geometry, we obtain the estimate

$$\|s_t\|_{\mathcal{C}^{k+2,\alpha}(\Sigma)} \leq C_{k,\alpha}(\|\rho_t\|_{\mathcal{C}^{k,\alpha}(\Sigma)} + 1)$$

where $C_{k,\alpha}$ is a t-independent constant. Therefore, passing to a subsequence, s_t converges to s_0 in $\mathcal{C}^\infty(\Sigma)$, which implies (i). The assertion (ii) follows from (i). □

4. The algebraic and analytic compactifications

4.1 The algebraic compactification of the Dolbeault moduli space

In this subsection, we present the algebraic method for compactifying the Dolbeault moduli space. This technique is based on the $\mathbb{C}^*$ action on $\mathcal{M}_{\mathrm{Dol}}$ and was introduced in [Sim97, Sch98, Hau98, dC21, KNPS15]. The gauge theoretic approach can be found in [Fan22a].

Theorem 4.1. *Let V be a complex algebraic variety with $\mathbb{C}^*$ action. Suppose that*

(i) the fixed point set of the $\mathbb{C}^*$ action is proper and that

(ii) for every $t \in \mathbb{C}^*$, $v \in V$, the limit $\lim_{t \to 0} t \cdot v$ exists.

Then the space $U := \{v \in V \mid \lim_{t \to \infty} t \cdot v \text{ does not exist}\}$ is open in V, and the quotient $U/\mathbb{C}^*$ is separated and proper.

We apply this to the Dolbeault moduli space. The first step is to note that the possible isotropy subgroups are limited.

Lemma 4.2. *Let $\xi = [(\mathcal{E}, \varphi)]$ be an $\mathrm{SL}(2,\mathbb{C})$ Higgs bundle equivalence class with $\mathcal{H}(\xi) \neq 0$. Then the stabiliser Γ_ξ of ξ for the $\mathbb{C}^*$ action is either trivial or $\mathbb{Z}/2$. The latter case holds if and only if $(\mathcal{E}, \varphi)$ and $(\mathcal{E}, -\varphi)$ are complex gauge equivalent.*

Proof. For $t \in \Gamma_\xi$, $\mathcal{H}(\xi) = \mathcal{H}(t \cdot \xi) = t^2 \mathcal{H}(\xi)$. Hence, $t^2 = 1$ if $\mathcal{H}(\xi) \neq 0$. □

By this lemma, the space $(\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0))/\mathbb{C}^*$ has an orbifold structure. In passing, we note that the fixed points of the $\mathbb{Z}/2$ action correspond to real representations under the nonabelian Hodge correspondence [Hit87a, Sec. 10].

By the properness of the Hitchin map $\mathcal{H}$ (see Theorem 2.1), it follows that $\lim_{t \to \infty} t \cdot \xi$ exists if and only if $\mathcal{H}(\xi) = 0$. Now define

$$\overline{\mathcal{M}}_{\mathrm{Dol}} = \left\{ (\mathcal{M}_{\mathrm{Dol}} \times \mathbb{C}^*) \coprod (\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0)) \right\} / \mathbb{C}^*. \tag{8}$$

The analytic topology on the disjoint union is generated by open sets $U \times W_1$ and

$$V \times (W_2 \cap \mathbb{C}^*) \amalg V \cap (\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0))$$

where $U, V \subset \mathcal{M}_{\mathrm{Dol}}$, $W_1, W_2 \subset \mathbb{C}$ are open, and $0 \notin W_1$, $0 \in W_2$. The topology on $\overline{\mathcal{M}}_{\mathrm{Dol}}$ is then the quotient topology, and it is straightforward to see that with this topology, it is compact.

Since $(\mathcal{M}_{\mathrm{Dol}} \times \mathbb{C}^*)/\mathbb{C}^* = \mathcal{M}_{\mathrm{Dol}}$, there is the natural inclusion

$$\iota : \mathcal{M}_{\mathrm{Dol}} \to \overline{\mathcal{M}}_{\mathrm{Dol}},\ \iota(\xi) = [(\xi, 1)]$$

where brackets denote the equivalence class under the $\mathbb{C}^*$ action. The boundary of $\overline{\mathcal{M}}_{\mathrm{Dol}}$ is

$$\partial\overline{\mathcal{M}}_{\mathrm{Dol}} = \overline{\mathcal{M}}_{\mathrm{Dol}} \setminus \iota(\mathcal{M}_{\mathrm{Dol}}) = (\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0))/\mathbb{C}^*.$$

There is also the *boundary map*

$$\iota_\partial : \mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0) \longrightarrow \partial\overline{\mathcal{M}}_{\mathrm{Dol}},\ \xi \mapsto [(\xi, 0)]$$

which is invariant under the $\mathbb{C}^*$ action; that is, $\iota_\partial(\lambda\xi) = \iota_\partial(\xi)$ for $\lambda \in \mathbb{C}^*$.

The $\mathbb{C}^*$ action on $\mathcal{M}_{\mathrm{Dol}}$ covers the square of the action on $\mathcal{B}$. Hence, it is natural to compactify $\mathcal{B}$ by projectivising:

$$\overline{\mathcal{B}} := \mathbb{P}(H^0(K^2) \oplus \mathbb{C}).$$

The inclusion is given, as usual, by

$$\iota_0 : \mathcal{B} \to \overline{\mathcal{B}},\ \iota(q) = [q \times \{1\}]$$

where $q \times \{1\} \in H^0(K^2) \oplus \mathbb{C}$. We also define $\partial\overline{\mathcal{B}} = \overline{\mathcal{B}} \setminus \iota_0(\mathcal{B}) \simeq \mathbb{P}(H^0(K^2))$, with boundary projection map

$$\iota_{0,\partial} : \mathcal{B} \setminus \{0\} \to \partial\overline{\mathcal{B}} \quad \iota_{0,\partial}(q) = [q \times \{0\}].$$

The Hitchin map $\mathcal{H} : \mathcal{M}_{\mathrm{Dol}} \to \mathcal{B}$ extends to $\overline{\mathcal{H}} : \overline{\mathcal{M}}_{\mathrm{Dol}} \to \overline{\mathcal{B}}$, where $\overline{\mathcal{H}}|_{\mathcal{M}_{\mathrm{Dol}}} := \iota_0 \circ \mathcal{H}$, and for every $[(\mathcal{E}, \varphi)]/\mathbb{C}^* \in \partial\overline{\mathcal{M}}_{\mathrm{Dol}}$,

$$\overline{\mathcal{H}}([(\mathcal{E}, \varphi)]/\mathbb{C}^*) := [(\mathcal{H}(\varphi), 0)] \subset \overline{\mathcal{B}}.$$

This is well-defined, since $\det(\varphi) \neq 0$ if $[(\mathcal{E}, \varphi)]/\mathbb{C}^* \in \partial\overline{\mathcal{M}}_{\mathrm{Dol}}$. Moreover,

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}} & \xrightarrow{\ \iota\ } & \overline{\mathcal{M}}_{\mathrm{Dol}} \\ \downarrow{\scriptstyle \mathcal{H}} & & \downarrow{\scriptstyle \overline{\mathcal{H}}} \\ \mathcal{B} & \xrightarrow{\ \iota_0\ } & \overline{\mathcal{B}} \end{array}$$

commutes.

There is a good algebraic structure on this compactification.

Theorem 4.3. *The compactified space $\overline{\mathcal{M}}_{\mathrm{Dol}}$ is a normal projective variety, and $\partial\overline{\mathcal{M}}_{\mathrm{Dol}}$ is a Cartier divisor of $\overline{\mathcal{M}}_{\mathrm{Dol}}$.*

The following characterisation of sequential convergence is useful: As $H^0(K^2)$ is a finite dimensional space, the L^2 norm on $q \in H^0(K_\Sigma^2)$ can be chosen arbitrarily, and we fix one such choice.

Proposition 4.4. *Let $[(\mathcal{E}_i, \varphi_i)] \in \mathcal{M}_{\mathrm{Dol}}$ be a sequence of Higgs bundles, and write $q_i = \det(\varphi_i)$ and $r_i = \|q_i\|_{L^2}^{\frac{1}{2}}$. Suppose $\limsup r_i \to \infty$. Then up to the subsequence:*

(i) there exists a Higgs bundle $[(\widehat{\mathcal{E}}_\infty, \hat{\varphi}_\infty)]$ with $\tilde{q}_\infty = \det(\tilde{\varphi}_\infty)$ and $\|\hat{q}_\infty\|_{L^2} = 1$ such that $\lim_{i\to\infty}[(\mathcal{E}_i, r_i^{-1}\varphi_i)] = [(\widehat{\mathcal{E}}_\infty, \hat{\varphi}_\infty)]$ in $\mathcal{M}_{\mathrm{Dol}}$ and $\lim_{i\to\infty} r_i^{-1} q_i = \hat{q}_\infty$ in $H^0(K^2)$;

(ii)

$$\lim_{i\to\infty} \iota[(\mathcal{E}_i,\varphi_i)] = \iota_\partial[(\widehat{\mathcal{E}}_\infty,\hat{\varphi}_\infty)] \text{ on } \overline{\mathcal{M}}_{\mathrm{Dol}} \textit{ and}$$
$$\lim_{i\to\infty} \iota_0(q_i) = \iota_{0,\partial}(\hat{q}_\infty) \text{ on } \overline{\mathcal{B}}.$$

Proof. The first point follows since the Hitchin map $\mathcal{H}$ is proper and since $\mathcal{H}(r_i^{-1}\varphi_i)$ is bounded. The second point follows directly from the definition. □

4.2 The analytic compactification of the Hitchin moduli space

We next describe the compactification of the Hitchin moduli space, as developed in [MSWW14, Moc16, Tau13a].

4.2.1 *Decoupled Hitchin equations.* We begin by defining the decoupled Hitchin equations. Recalling the notation from Section 2.4, let E be a trivial, smooth, rank 2 vector bundle over a Riemann surface Σ, and let H_0 be a background Hermitian metric on E. Let Z be a finite set of distinct points in Σ. For a smooth unitary connection A on $E|_{\Sigma\setminus Z}$ and a smooth $\phi=\varphi+\varphi^\dagger \in \Omega^1(i\mathfrak{su}(E))|_{\Sigma\setminus Z}$, the *decoupled Hitchin equations* on $\Sigma\setminus Z$ are:

$$F_A=0 \;\; [\varphi,\varphi^\dagger]=0 \;\; \bar{\partial}_A\varphi=0. \tag{9}$$

Solutions to (9) alone may be quite singular near Z, so we make the following restriction:

Definition 4.5. *A solution (A,ϕ) to (9) is called* admissible *if $\phi\neq 0$ and if $|\phi|$ extends to a continuous function on Σ with $|\phi|^{-1}(0)=Z$.*

By a *limiting configuration*, we always mean an admissible solution to the decoupled Hitchin equations. Clearly, Z is determined by (A,ϕ). Admissibility guarantees that $\det(\varphi)$ extends to a holomorphic quadratic differential $q=\det(\varphi)$ on Σ, with $Z=q^{-1}(0)$ being the zero locus. Hence, the spectral curve S_q is well-defined. We emphasize that Z may vary for different admissible solutions, but one always has $\#Z\leq 4g-4$.

The equivalence relation on limiting configurations is that $(A_1,\phi_1)\sim(A_2,\phi_2)$ if $Z_1=Z_2$ and if $(A_1,\phi_1)g=(A_2,\phi_2)$ for a smooth unitary gauge transformation g on $\Sigma\setminus Z_1$. The moduli space of decoupled Hitchin equations is then

$$\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}=\{\text{admissible solutions to (9)}\}/\sim.$$

We denote by $\mathcal{M}_{\mathrm{Hit},q}^{\mathrm{Lim}}$ the elements in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$, with the determinant of the Higgs field equal to a quadratic differential q. In this case, the equivalence relation is induced by the action of the unitary gauge group over $\Sigma\setminus Z$, $Z=q^{-1}(0)$.

There is a natural $\mathbb{C}^*$ action on the moduli space $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$: Given $(A,\phi=\varphi+\varphi^\dagger)\in\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$ and $t\in\mathbb{C}^*$, we set $t\cdot[(A,\phi)]=[(A,t\varphi+\bar{t}\varphi^\dagger)]$, which is also a solution to (9).

4.2.2 *Compactification of the Hitchin moduli space.* The following compactness result is due to Taubes [Tau13b] and Mochizuki [Moc16] (see also [He20]).

Proposition 4.6. *Let (A_i,φ_i) be a sequence of solutions to (5), with $q_i=\det(\varphi_i)\in H^0(K^2)$. Then*

(i) if $\limsup \|q_i\|_{L^2(\Sigma)}<\infty$, then there is a subsequence (also denoted $\{i\}$), a smooth solution (A_∞,ϕ_∞) to (5), and a sequence g_i of smooth unitary gauge transformations on Σ such that $(A_i,\phi_i)g_i$ converges smoothly to (A_∞,ϕ_∞) on Σ;

(ii) if $\lim \|q_i\|_{L^2(\Sigma)} = \infty$*, then there is a subsequence (also denoted* $\{i\}$*), and* $q_\infty \in H^0(K^2)$*, leading to*

$$\frac{q_i}{\|q_i\|_{L^2}} \longrightarrow q_\infty$$

over Σ*, and an admissible solution* $(A_\infty, \phi_\infty = \varphi_\infty + \varphi_\infty^\dagger)$ *to (9), with* $Z_\infty := q_\infty^{-1}(0)$ *and with smooth unitary gauge transformations* g_i *on* $\Sigma \setminus Z_\infty$ *such that over any open set* $\Omega \Subset \Sigma \setminus Z_\infty$*,* $(A_i)g_i \to A_\infty$*, and*

$$\frac{g_i^{-1}\phi_i g_i}{\|\phi\|_{L^2}} \longrightarrow \phi_\infty$$

smoothly on Ω*.*

There is also a compactness result for sequences of solutions in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$.

PROPOSITION 4.7. *Let* $[(A_i, \phi_i = \varphi_i + \varphi_i^\dagger)] \in \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$ *be a sequence of admissible solutions to (9), and let* $q_i = \det(\varphi_i)$ *be the corresponding quadratic differentials. Then, after passing to a subsequence, there are* $t_i \in \mathbb{C}^*$*, a limiting configuration* $(A_\infty, \phi_\infty = \varphi_\infty + \varphi_\infty^\dagger)$ *with quadratic differential* $q_\infty = \det(\varphi_\infty) \neq 0$*, and a sequence* g_i *of smooth gauge transformations on* $\Sigma \setminus Z_\infty$*, such that:*

(i) $t_i^2 q_i$ *converges smoothly to* q_∞ and

(ii) over any open set $\Omega \Subset X \setminus Z_\infty$*,* $(A_i, t_i \cdot \phi_i)g_i$ *converges smoothly to* (A_∞, ϕ_∞)*.*

Proof. Write $q_i = \det(\varphi_i) \in H^0(K^2)$. Adjusting by t_i if necessary, we may assume that q_i converges to q_∞ over Σ. Also, since $F_{A_i} = 0$ over $\Sigma \setminus Z_i$ and Z_i converges to Z_∞, we can apply both Uhlenbeck compactness and the classical bootstrapping method to obtain A_∞ such that up to gauge A_i, it converges smoothly to A_∞ over $\Sigma \setminus Z_\infty$. Finally, the convergence of φ_i follows by the bound on q_i's. □

4.2.3 *The topology on the compactified space.* We now carefully define the topology on the space $\mathcal{M}_{\mathrm{Hit}} \coprod \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$. Choose a metric in the conformal class on Σ. Let $W^{k,2}$ denote the Sobolev spaces on Σ of distributional sections with at least k derivatives in L^2. For a finite set of points $Z \subset \Sigma$ (or indeed any closed subset),

$$W_{\mathrm{loc}}^{k,2}(\Sigma \setminus Z) := \{f \mid f \in W^{k,2}(K),\ K \subset \Sigma \setminus Z,\ K \text{ compact}\}.$$

These definitions extend easily to the space of connections and $\Omega^1(i\mathfrak{su}(E))$ for a Hermitian vector bundle (E, H_0) over Σ with a fixed, smooth background connection.

Let ω_n be a nested collection of open sets with $\omega_n \subset \overline{\omega_n} \subset \omega_{n+1}$ and with $\bigcup_n \omega_n = \Sigma \setminus Z$. We then define the seminorms $\|f\|_n := \|f\|_{W^{k,2}(\omega_n)}$; in terms of these, $W_{\mathrm{loc}}^{k,2}(\Sigma \setminus Z)$ is a Fréchet space.

For any $q \in H^0(K^2) \setminus \{0\}$, set $Z_q := q^{-1}(0)$, and consider the moduli space

$$\mathbb{M}_q = \Big\{[(A,\phi)] \in \mathcal{M}_{\mathrm{Hit},q^*} \cap W^{k,2}(\Sigma)\Big\} \cup \Big\{[(A,\phi)] \in \mathcal{M}_{\mathrm{Hit},q}^{\mathrm{Lim}} \cap W_{\mathrm{loc}}^{k,2}(\Sigma \setminus Z_q)\Big\}/\mathbb{C}^*.$$

Here we give a more precise explanation of the above notation. The space $\mathcal{M}_{\mathrm{Hit},q^*}$ consists of solutions $(A, \phi = \varphi + \varphi^\dagger)$ to the Hitchin equations such that $\det(\varphi) = tq$ for some nonzero complex number t. Moreover, the notation $[(A,\phi)] \in \mathcal{M}_{\mathrm{Hit},q^*} \cap W^{k,2}(\Sigma)$ refers to the equivalence class of $(A,\phi) \in W^{k,2}(\Sigma)$ modulo unitary gauge transformations in $W^{k+1,2}(\Sigma)$. Similarly, $[(A,\phi)] \in \mathcal{M}_{\mathrm{Hit},q}^{\mathrm{Lim}} \cap W^{\mathrm{loc}}$ consists of the equivalence class of $(A,\phi) \in W_{\mathrm{loc}}^{k,2}(\Sigma \setminus Z_q)$ modulo unitary gauge transformations in $W^{k+1,2}_{\mathrm{loc}}(\Sigma \setminus Z_q)$, and the $\mathbb{C}^*$ action is given by $t \cdot [(A,\phi)] \to [(A, t\phi)]$. By classical bootstrapping of the gauge-theoretic elliptic equations, $\mathbb{M}_q$ is independent of $k \geq 2$.

Next define $\mathbb{M} := \mathcal{M}_0 \cup \bigcup_{q \in H^0(K^2)\setminus\{0\}} \mathbb{M}_q$, and based on the definition, we have $\mathbb{M} = \mathcal{M}_{\mathrm{Hit}} \cup \mathcal{M}^{\mathrm{Lim}}_{\mathrm{Hit}}/\mathbb{C}^*$. Its topology is generated by two types of open sets. For interior points $\xi = [(A,\phi)] \in \mathcal{M}_{\mathrm{Hit}} \subset \mathbb{M}$, we use the open sets

$$V_{\xi,\epsilon} := \left\{ [(A',\phi')] \in \mathcal{M}_{\mathrm{Hit}} \mid \|A' - A\|_{W^{k,2}(\Sigma)} + \|\phi' - \phi\|_{W^{k,2}(\Sigma)} < \epsilon \right\}$$

from the topology of $\mathcal{M}_{\mathrm{Hit}}$. For any boundary point $\xi_0 \in \mathcal{M}^{\mathrm{Lim}}_{\mathrm{Hit}}/\mathbb{C}^*$, choose a representative (A_0,ϕ_0) with $\|\phi_0\|_{L^2} = 1$. Let $q = \det(\phi_0)$, and fix any open set $\omega \Subset \Sigma \setminus Z_q$. Then, setting $\mathcal{M}^*_{\mathrm{Hit}} = \mathcal{M}_{\mathrm{Hit}} \setminus \mathcal{H}^{-1}(0)$,

$$\begin{aligned} U_{\xi_0,\omega,\epsilon} := &\left\{ (A,\phi) \in \mathcal{M}^*_{\mathrm{Hit}} \mid \|A - A_0\|_{W^{k,2}(\omega)} + \inf_{\theta \in S^1} \| \|\phi\|_{L^2}^{-\frac{1}{2}} \phi - e^{i\theta}\phi_0 \|_{W^{k,2}(\omega)} < \epsilon,\ \|\phi\|_{L^2} > \frac{1}{\epsilon} \right\} \\ &\cup \left\{ (A,\phi) \in \mathcal{M}^{\mathrm{Lim}}_{\mathrm{Hit}} | \|A - A_0\|_{W^{k,2}(\omega)} + \|\phi - \phi_0\|_{W^{k,2}(\omega)} < \epsilon \right\} \end{aligned}$$

defines an open set around ξ_0. The sets $U_{\xi_0,\omega,\epsilon}$ and $V_{\xi,\epsilon}$ generate the topology on $\mathbb{M}$.

Theorem 4.8. *The space $\mathbb{M}$ is Hausdorff and compact.*

Proof. The Hausdorff property follows from the definition of the topology. By Propositions 4.6 and 4.7, $\mathbb{M}$ is sequentially compact. Moreover, using this explicit base for the topology, $\mathbb{M}$ is first countable and, hence, compact. □

We may now define the compactification of the Hitchin moduli space as the closure $\overline{\mathcal{M}}_{\mathrm{Hit}} \subset \mathbb{M}$; we write $\partial\overline{\mathcal{M}}_{\mathrm{Hit}}$ for the boundary of the closure and $\overline{\mathcal{M}}_{\mathrm{Hit},q^*} := \overline{\mathcal{M}}_{\mathrm{Hit}} \cap \mathbb{M}_q$ for the subset of elements with a fixed quadratic differential.

The following result is described in [MSWW16, OSWW20, MSWW19].

Theorem 4.9. *If q has only simple zeros, then $\overline{\mathcal{M}}_{\mathrm{Hit},q^*} = \mathbb{M}_q$.*

In other words, the compactification of any slice where q does not lie in the discriminant locus is 'the obvious one'.

5. Parabolic modules and stratification of BNR data

In this section, we review the notion of a parabolic module, as described in [Reg80, Coo93, Coo98, GO13]. This concept leads to a partial normalisation of the generalised Jacobian and Prym varieties of the spectral curve.

5.1 Normalization of the spectral curve

Let $q \neq 0$ be a quadratic differential with an irreducible, singular spectral curve $S = S_q$. The zeros of q define the divisor $\mathrm{Div}(q) = \sum_{i=1}^{r_1} m_i p_i + \sum_{j=1}^{r_2} n_j p'_j$, where the m_i and n_j are even and odd integers, respectively; and, hence, r_1 and r_2 are the numbers of even and odd zeros, respectively, counted without multiplicity. Write $Z_{\mathrm{even}} = \{p_1, \ldots, p_{r_1}\}$, $Z_{\mathrm{odd}} = \{p'_1, \ldots, p'_{r_2}\}$ and $Z = Z_{\mathrm{even}} \cup Z_{\mathrm{odd}}$; so $\#Z = r = r_1 + r_2$.

The map $\pi : S \to \Sigma$ is a double covering branched along Z; hence, we may view p_i and p'_i as points in S. For $x \in S$, let $\mathcal{O}_x$ be the algebraic local ring, $\mathcal{O}^*_x$ its group of units and R_x the completion. We say that S has an A_n singularity at x if $R_x \cong \mathbb{C}[[r,s]]/(r^2 - s^{n+1})$, where $n \geq 1$.

If S has an A_1 singularity at x, we call it a *nodal* singularity, and if S has an A_2 singularity at x, we call it a *cusp* singularity.

Let $p: \widetilde{S} \to S$ be the normalisation of S, and let $\tilde{\pi} := \pi \circ p$:

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{p} & S \\ & \searrow^{\tilde{\pi}} & \downarrow \pi \\ & & \Sigma \end{array} \tag{10}$$

For even zeros p_i, we write $p^{-1}(p_i) = \{\tilde{p}_i^+, \tilde{p}_i^-\}$, and for odd zeros p'_i we write $p^{-1}(p'_i) = \tilde{p}'_i$. Since $\pi: S \to \Sigma$ is a branched double cover, the involution σ on S extends to an involution of $\widetilde{S}$, which we also denote by σ. Note that $\sigma(\tilde{p}'_i) = \tilde{p}'_i$ while $\sigma(\tilde{p}_i^{\pm}) = \tilde{p}_i^{\mp}$.

The ramification divisor $\Lambda' = \frac{1}{2}\sum_{i=1}^{r_1} m_i p_i + \frac{1}{2}\sum_{j=1}^{r_2}(n_j - 1)p'_j$, is a (Weil) divisor on S, and there is an exact sequence:

$$0 \longrightarrow \mathcal{O}_S \longrightarrow p_*\mathcal{O}_{\widetilde{S}} \longrightarrow \sum_{x \in \mathrm{Supp}(\Lambda')} \widetilde{\mathcal{O}}_x/\mathcal{O}_x \longrightarrow 0. \tag{11}$$

The genus of $\widetilde{S}$ is $g(\widetilde{S}) = 4g - 3 - \deg(\Lambda') = 2g - 1 + r_2/2$.

5.2 Jacobian under the pull-back to the normalization

We now recall some facts about the Jacobian under the pull-back to the normalization (compare to [GO13]). Let $x \in Z \subset S$ be a singular point; that is, either $x \in Z_{\text{even}}$ or $x = p'_j$ with $n_j \geq 3$. Let $\widetilde{\mathcal{O}}_x$ be the integral closure of $\mathcal{O}_x$. Set $V := \prod_{x \in Z} \widetilde{\mathcal{O}}_x^*/\mathcal{O}_x^*$. Then we have the following well-known short exact sequence:

$$0 \longrightarrow V \longrightarrow \mathrm{Jac}(S) \xrightarrow{p^*} \mathrm{Jac}(\widetilde{S}) \longrightarrow 0. \tag{12}$$

This will play an important role later on.

5.2.1 *Hitchin fibre.* We examine the locally free part $\mathcal{T}$ of the Hitchin fibre under the pull-back. Here, $\mathcal{T}$ is defined to be the set of $L \in \mathrm{Pic}^{2g-2}(S)$ such that $\det(\pi_* L) = \mathcal{O}_\Sigma$ [see (4)]. Although Λ' is a divisor S, it could also be considered to be a divisor on Σ by the identification of p_i, p'_j and $\pi(p_i), \pi(p'_j)$. To shorten the notation, we write $\mathcal{O}_\Sigma(\Lambda')$ for the corresponding line bundle on Σ. For any $L \in \mathrm{Pic}(S)$, from (11) we see that $\det(\tilde{\pi}_* p^* L) \cong \det(\pi_* L) \otimes \mathcal{O}_\Sigma(\Lambda')$. We define a new set, $\widetilde{\mathcal{T}}$, as follows:

$$\widetilde{\mathcal{T}} := \{\widetilde{L} \in \mathrm{Pic}^{2g-2}(\widetilde{S}) \mid \det(\tilde{\pi}_* L) \cong \mathcal{O}(\Lambda')\}.$$

Then p^* maps $\mathcal{T}$ to $\widetilde{\mathcal{T}}$. Furthermore, if $L_1, L_2 \in \mathrm{Pic}(S)$ satisfy $p^* L_1 \cong p^* L_2$, then we have $\pi_* L_1 \cong \pi_* L_2$. This means that the fibre of $p^*: \mathrm{Jac}(S) \to \mathrm{Jac}(\widetilde{S})$ is the same as that of $p^*: \mathcal{T} \to \widetilde{\mathcal{T}}$, resulting in the following fibration:

$$V \longrightarrow \mathcal{T} \xrightarrow{p^*} \widetilde{\mathcal{T}}. \tag{13}$$

5.3 Torsion-free sheaves

Now we present Cook's parametrisation of rank 1 torsion-free sheaves on curves with Gorenstein singularities (see [Coo98, p. 40] and [Coo93, Reg80]). An explicit computation of the invariants used in this article is provided in Appendix A. Let $x \in Z$ be a singular point of S, and let

$L \to S$ be a rank 1 torsion-free sheaf. We again let $\mathcal{O}_x$ denote the local ring at x, $\widetilde{\mathcal{O}}_x$ its integral closure, and $\delta_x = \dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_x/\mathcal{O}_x)$. According to [GP93, Lemma 1.1], there exists a fractional ideal I_x that is isomorphic to L_x and uniquely defined up to multiplication by a unit of $\widetilde{\mathcal{O}}_x$ such that $\mathcal{O}_x \subset I_x \subset \widetilde{\mathcal{O}}_x$. We define $\ell_x := \dim_{\mathbb{C}}(I_x/\mathcal{O}_x)$ and $b_x := \dim_{\mathbb{C}}(T(I_x \otimes_{\mathcal{O}_x} \widetilde{\mathcal{O}}_x))$, where T means the torsion subsheaf. Then ℓ_x and b_x are invariants of L.

Let $\mathcal{K}_x$ be the field of fractions of $\mathcal{O}_x$. The *conductor* of $I_x \subset \widetilde{\mathcal{O}}_x$ is defined to be

$$C(I_x) = \{u \in \mathcal{K}_x \mid u \cdot \widetilde{\mathcal{O}}_x \subset I_x\}.$$

The singularity is characterized by the following dimensions:

$$\underbrace{C(\mathcal{O}_x) \subset \overbrace{C(I_x) \subset \overbrace{\mathcal{O}_x \subset \underbrace{I_x \subset \widetilde{\mathcal{O}}_x}_{\delta_x-\ell_x}}^{\delta_x}}^{2\delta_x-2\ell_x}}_{2\delta_x}. \tag{14}$$

For $x = p_i \in Z_{\text{even}}$, we have $\delta_{p_i} = m_i/2$, and there are two maximal ideals $\mathfrak{m}_\pm$ in $\widetilde{\mathcal{O}}_x$ corresponding to the points $\tilde{p}_i^\pm$. We let $(\widetilde{\mathcal{O}}_{p_i}/C(I_{p_i}))_{\mathfrak{m}_\pm}$ be the localization by the ideals $\mathfrak{m}_\pm$, and we define $a_{\tilde{p}_i^\pm} := \dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_{p_i}/C(I_{p_i}))_{\mathfrak{m}_\pm}$. Moreover, we have $\dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_{p_i}/C(\mathcal{O}_{p_i}))_{\mathfrak{m}_\pm} = m_i/2 = \delta_{p_i}$. By Appendix A, $a_{\tilde{p}_i^\pm} = (m_i/2) - \ell_{p_i}$, and therefore $a_{\tilde{p}_i^+} + a_{\tilde{p}_i^-} = 2\delta_{p_i} - 2\ell_{p_i}$ and also $b_{p_i} = \ell_{p_i}$. Define

$$V(L_{p_i}) = \{(c_i^+, c_i^-) \mid c_i^\pm \in \mathbb{Z}_{\geq 0} \;\; c_i^+ + c_i^- = \ell_{p_i}\}.$$

For $x = p_i' \in Z_{\text{odd}}$, we have $\delta_{p_i'} = (n_i - 1)/2$, and the maximal ideal $\mathfrak{m}$ of $\widetilde{\mathcal{O}}_x$ is unique. We define $a_{\tilde{p}_i'} := \dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_{p_i'}/C(I_{p_i'}))_{\mathfrak{m}}$. By Appendix A, we have $a_{\tilde{p}_i'} = 2\delta_{p_i'} - 2\ell_{p_i'}$ and $b_{p_i'} = \ell_{p_i'}$. Moreover, $\dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_{p_i'}/C(\mathcal{O}_{p_i'}))_{\mathfrak{m}} = n_i - 1 = 2\delta_{p_i'}$. In this case we set $V(L_{p_i'}) = \{\ell_{p_i'}\}$.

Let $\eta : \widetilde{\mathcal{O}}_x \to \widetilde{\mathcal{O}}_x/C(\mathcal{O}_x)$ be the quotient map. We define

$$S(L_x) := \{\mathcal{O}_x\text{-submodules } F_x \subset \widetilde{\mathcal{O}}_x/C(\mathcal{O}_x) \mid \dim_{\mathbb{C}}(F_x) = \delta_x \;\; \eta^{-1}(F_x) \cong L_x\}.$$

Hence, if $J_x = \eta^{-1}(F_x)$ with $F_x \in S(L_x)$, there exists an ideal $\mathfrak{t}_x$ in $\widetilde{\mathcal{O}}_x$ such that $J_x = \mathfrak{t}_x \cdot L_x$. For $x = p_i \in Z_{\text{even}}$, we obtain two integers: $c_i^\pm = \dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_x/(\mathfrak{t}_x \cdot \widetilde{\mathcal{O}}_x))_{\mathfrak{m}_\pm}$. By [Coo98, Lemma 6], $(c_i^+, c_i^-) \in V(L_{p_i})$, for $x = p_i' \in Z_{\text{odd}}$ and $\dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_x/(\mathfrak{t}_x \cdot \widetilde{\mathcal{O}}_x)) = \ell_{p_i'} \in V(L_{p_i'})$, and these only depend only on F_x. Hence, there is a well-defined map:

$$\kappa_x : S(L_x) \longrightarrow V(L_x) \;:\; \begin{cases} F_x \to (c_i^+, c_i^-) & \text{when } x = p_i, \\ F_x \to \ell_{p_i'} & \text{when } x = p_i'. \end{cases}$$

Lemma 5.1. *For $x \in Z$, the connected components of $S(L_x)$ are parameterized by elements in $V(L_x)$.*

Set $V(L) := \prod_{x \in Z} V(L_x)$ and $S(L) := \prod_{x \in Z} S(L_x)$. Write $N(L) := |V(L)|$ for the number of points in $V(L)$. There is a map

$$\kappa := \prod_{x \in Z} \kappa_x : S(L) \longrightarrow V(L).$$

For any $\mathbf{c} \in V(L)$, write $\mathbf{c} = (c_1^{\pm}, \ldots, c_{r_1}^{\pm}, \ell_{p_1'}, \ldots, \ell_{p_{r_2}'})$. Associate to $\mathbf{c}$ the divisor

$$D_{\mathbf{c}} = \sum_{i=1}^{r_1} (c_i^+ \tilde{p}_i^+ + c_i^- \tilde{p}_i^-) + \sum_{i=1}^{r_2} \ell_{p_i'} \tilde{p}_i'$$

on $\widetilde{S}$. Composing κ with the map above, we define

$$\varkappa : S(L) \longrightarrow \mathrm{Div}(\widetilde{S}) \ : \ \prod_{x \in Z} F_x \mapsto \mathbf{c} \mapsto D_{\mathbf{c}}. \tag{15}$$

The following result is straightforward but important:

Proposition 5.2. *L is locally free if and only if $\varkappa = 0$ on $S(L)$.*

Proof. L is locally free if and only if $\ell_x = 0$ for all $x \in Z$. The claim then follows directly from the definition of $D_{\mathbf{c}}$. □

5.4 Parabolic modules

In this subsection, we define the notion of a parabolic module, following [Reg80, Coo93, Coo98]. First note that $\dim_{\mathbb{C}}(\widetilde{\mathcal{O}}_x / C(\mathcal{O}_x)) = 2\delta_x$ (compare to (14)). Let $\mathrm{Gr}(\delta_x, \widetilde{\mathcal{O}}_x / C(\mathcal{O}_x))$ be the Grassmannian of δ_x dimensional subspaces of the vector space $\widetilde{\mathcal{O}}_x / C(\mathcal{O}_x)$. Then $\widetilde{\mathcal{O}}_x^*$ acts on $\mathrm{Gr}(\delta_x, \widetilde{\mathcal{O}}_x / C(\mathcal{O}_x))$ by multiplication, with fixed points corresponding to δ_x-dimensional $\mathcal{O}_x$ submodules of $\widetilde{\mathcal{O}}_x / C(\mathcal{O}_x)$. We write $\mathcal{P}(x)$ for the (reduced) variety of fixed points. This is a closed subvariety of $\mathrm{Gr}(\delta_x, \widetilde{\mathcal{O}}_x / C(\mathcal{O}_x))$.

Suppose x is an A_n singularity. For notational convenience, we write $\mathcal{P}(A_n) := \mathcal{P}(x)$. We have the following:

Proposition 5.3. *The following holds:*

(i) $\mathcal{P}(A_n)$ is connected and depends only on δ_x. Also, $\dim_{\mathbb{C}} \mathcal{P}(A_{2n}) = n$, and we have isomorphisms $\mathcal{P}(A_{2n-1}) \cong \mathcal{P}(A_{2n})$.

(ii) If $P(A_0)$ is defined to be a point, then the inclusions $\mathcal{P}(A_0) \subset \mathcal{P}(A_2) \subset \cdots \subset \mathcal{P}(A_{2n})$ give a cell decomposition of $\mathcal{P}(A_{2n})$.

(iii) The singular locus $\mathrm{Sing}(\mathcal{P}(A_{2n})) \cong \mathcal{P}(A_{2n-4})$. In particular, it has codimension ≥ 2. Moreover, $\mathcal{P}(A_1) = \mathcal{P}(A_2) \cong \mathbb{C}P^1$, and $\mathcal{P}(A_4)$ is a quadric cone.

Define $\mathscr{P}(S) = \prod_{x \in Z} \mathcal{P}(x)$. This depends only on the curve singularity of S. Let $J \in \mathrm{Pic}(\widetilde{S})$. As vector spaces,

$$J_{\tilde{p}_i^+}^{\oplus \frac{m_i}{2}} \oplus J_{\tilde{p}_i^-}^{\oplus \frac{m_i}{2}} \cong \widetilde{\mathcal{O}}_{p_i} / C(\mathcal{O}_{p_i}) \quad J_{\tilde{p}_i'}^{\oplus (n_i - 1)} \cong \widetilde{\mathcal{O}}_{p_i'} / C(\mathcal{O}_{p_i'}).$$

Definition 5.4. *A parabolic module $\mathrm{PMod}(\widetilde{S})$ consists of pairs (J, v), where $J \in \mathrm{Jac}(\widetilde{S})$ and $v = \prod_{x \in Z} v_x$, with $v_x \in \mathcal{P}(x)$.*

By [Coo98, p. 41], $\mathrm{PMod}(\widetilde{S})$ has a natural algebraic structure. Let $\mathrm{pr} : \mathrm{PMod}(\widetilde{S}) \to \mathrm{Jac}(\widetilde{S})$ be the projection to the first component. Then pr defines a fibration of $\mathrm{PMod}(\widetilde{S})$ with fibre $\mathscr{P}(S)$. Moreover, there is a finite morphism $\tau : \mathrm{PMod}(\widetilde{S}) \to \overline{\mathrm{Jac}}(S)$ defined by sending $(J, v) \to L$, where L is given by:

$$0 \longrightarrow L \longrightarrow p_* J \longrightarrow (J \otimes \mathcal{O}_\Lambda)/v \longrightarrow 0.$$

There is a corresponding diagram:

$$\begin{array}{ccccc} \mathscr{P}(S) & \longrightarrow & \mathrm{PMod}(\widetilde{S}) & \xrightarrow{\mathrm{pr}} & \mathrm{Jac}(\widetilde{S}) \\ & & \big\downarrow{\scriptstyle\tau} & & \\ & & \overline{\mathrm{Jac}}(S) & & \end{array}$$

The map τ may be regarded as the compactification of the pull-back normalization map p^* in (12).

Theorem 5.5 [Coo98, Thm. 1].
For the map $\tau : \mathrm{PMod}(\widetilde{S}) \to \overline{\mathrm{Jac}}(S)$ defined above,

(i) *τ is a finite morphism, where the fibre over L consists of $N(L)$ points.*
(ii) *The restriction $\tau : \tau^{-1}\mathrm{Jac}(S) \to \mathrm{Jac}(S)$ is an isomorphism. Moreover, for $L \in \mathrm{Jac}(S)$, we have $\mathrm{pr} \circ \tau^{-1}(L) = p^*(L)$.*
(iii) *Suppose $\tau(J, v) = L$. For $x \in Z$, we have $v_x \in S(L_x)$. Let $D_v = \varkappa(v)$ be the divisor defined in (15). Then*

$$0 \longrightarrow p^*L/T(p^*L) \longrightarrow J \longrightarrow J \otimes \mathcal{O}_{D_v} \longrightarrow 0.$$

In particular, $p^*L/T(p^*L) = J(-D_v)$.

Suppose that all of the zeros of the quadratic differential q are odd. Then for $L \in \overline{\mathrm{Jac}}(S)$, $N(L) = 1$, and we can deduce the following:

Corollary 5.6. *If $q^{-1}(0) = \{p'_1, \dots, p'_r\}$ and all zeroes have odd multiplicity, then $\tau : \mathrm{PMod}(\widetilde{S}) \to \overline{\mathrm{Jac}}(S)$ is a bijection. Moreover, for $L \in \overline{\mathrm{Jac}}(S)$ with $\tau(J, v) = L$, we have*

$$p^*L/T(p^*L) = J\Big(-\sum \ell_{p'_i}\tilde{p}'_i\Big).$$

For convenience, we recall the canonical example of a parabolic module.

Example 5.7. Suppose q contains $4g-2$ simple zeros and one zero x of order 2. Then the spectral curve S has one nodal singularity at x. Denote $p : \widetilde{S} \to S$ as the normalization, with $p^{-1}(x) = \{\tilde{x}_+, \tilde{x}_-\}$. Then $\mathscr{P}(S) = \mathbb{C}P^1$, and we obtain a fibration $\mathbb{C}P^1 \to \mathrm{PMod}(\widetilde{S}) \to \mathrm{Jac}(\widetilde{S})$. Let $L \in \overline{\mathrm{Jac}}(S) \setminus \mathrm{Jac}(S)$. If we write $\widetilde{L} := p^*L/T(p^*L)$, then

$$\tau^{-1}(L) = \{(\widetilde{L} \otimes \mathcal{O}(\tilde{x}_+), v_+), (\widetilde{L} \otimes \mathcal{O}(\tilde{x}_-), v_-)\}.$$

We can define two sections:

$$s_\pm : \mathrm{Jac}(\widetilde{S}) \longrightarrow \mathrm{PMod}(\widetilde{S}) : J \mapsto (J, v_\pm)$$

where $v_+ = [1, 0]$, $v_- = [0, 1]$. Then $\overline{\mathrm{Jac}}(S)$ is the quotient of $\mathrm{PMod}(\widetilde{S})$ given by the identification

$$\overline{\mathrm{Jac}}(S) \cong \mathrm{PMod}(\widetilde{S})/(s_+ \sim \mathcal{O}(\tilde{x}_-\tilde{x}_+)s_-).$$

In particular, $\mathrm{PMod}(\widetilde{S})$ is not a fibration over $\overline{\mathrm{Jac}}(S)$.

Proposition 5.8. *The singular set of $\mathrm{PMod}(\widetilde{S})$ has codimension at least 2. Moreover, if the spectral curve S contains only cusp or nodal singularities, then $\mathrm{PMod}(\widetilde{S})$ is smooth.*

Proof. As the singularities of $\mathrm{PMod}(\widetilde{S})$ come from the space $\mathscr{P}(S)$, the claim follows from Proposition 5.3. □

Let $\mathcal{P} := \{L \in \mathrm{Jac}(S) \mid \det(\pi_* L) \cong K^{-1}\}$ and $\overline{\mathcal{P}}$ be the closure of $\mathcal{P}$ in $\overline{\mathrm{Pic}}(S)$. Since we focus on $\mathrm{SL}(2,\mathbb{C})$ Higgs bundles, we must consider the parabolic module compactification of the fibration

$$0 \longrightarrow V \longrightarrow \mathcal{P} \xrightarrow{p^*} \mathrm{Prym}(\widetilde{S}/\Sigma) \longrightarrow 0.$$

Setting $\widehat{\mathrm{PMod}}(\widetilde{S}) := \tau^{-1}(\overline{\mathcal{P}})$, there is a diagram from [GO13, p. 17]

$$\begin{array}{ccccc} \mathscr{P}(S) & \longrightarrow & \widehat{\mathrm{PMod}}(\widetilde{S}) & \xrightarrow{\mathrm{pr}} & \mathrm{Prym}(\widetilde{S}/\Sigma) \\ & & \big\downarrow \tau & & \\ & & \overline{\mathcal{P}} & & \end{array} \tag{16}$$

Theorem 5.5 proves that $\mathrm{pr} \circ \tau^{-1}|_{\mathcal{P}} = p^*$.

5.5 Stratifications of the BNR data

Recall that $\overline{\mathcal{T}}$ (resp. $\overline{\mathcal{P}}$) is the natural compactification of $\mathcal{T}$ (resp. $\mathcal{P}$) induced by the inclusion $\mathrm{Pic}(S) \subset \overline{\mathrm{Pic}}(S)$. Parabolic modules define a stratification of $\overline{\mathcal{P}}$ and $\overline{\mathcal{T}}$. In the following, $\pi : S \to \Sigma$ is a branched double cover, σ is the associated involution on S, and by σ we also denote its extension to an involution on the normalization $\widetilde{S}$ of S.

For a rank 1 torsion-free sheaf $L \in \overline{\mathrm{Pic}}(S)$, consider the map

$$p_{\mathrm{tf}}^{\star} : \overline{\mathrm{Pic}}(S) \longrightarrow \mathrm{Pic}(\widetilde{S}) \;\; p_{\mathrm{tf}}^{\star}(L) := p^* L / T(p^* L)$$

that is, the torsion-free part of the pull-back to the normalization. By [Rab79], $p_{\mathrm{tf}}^{\star}(L) = p^* L$ at $x \in \widetilde{S}$ if and only if L is locally free at $p(x) \in S$.

Using the previous conventions, recall that we have the divisor

$$\Lambda = \sum_{i=1}^{r_1} \frac{m_i}{2} (\tilde{p}_i^+ + \tilde{p}_i^-) + \sum_{j=1}^{r_2} n_j \tilde{p}_j'$$

on $\widetilde{S}$.

Definition 5.9. *An effective divisor $D \in \mathrm{Div}(\widetilde{S})$ is called a σ-divisor if*

(i) $D \leq \Lambda$ and $\sigma^* D = D$;

(ii) and for any $x \in \mathrm{Fix}(\sigma)$, $D|_x = d_x x$, where $d_x \equiv 0 \mod 2$.

The σ-divisors play an important role in describing the singular Hitchin fibres.

Proposition 5.10. *Let $L \in \overline{\mathcal{P}}$, and write $\widetilde{L} := p_{\mathrm{tf}}^{\star} L$. Then we have $\widetilde{L} \otimes \sigma^* \widetilde{L} = \mathcal{O}(-D)$ for D a σ-divisor.*

For a σ-divisor D, define

$$\widetilde{\mathcal{T}}_D = \{J \in \mathrm{Pic}(\widetilde{S}) \mid J \otimes \sigma^* J = \mathcal{O}(\Lambda - D)\} \;;$$
$$\widetilde{\mathcal{P}}_D = \{J \in \mathrm{Pic}(\widetilde{S}) \mid J \otimes \sigma^* J = \mathcal{O}(-D)\}. \tag{17}$$

By [Hor22a, Prop. 5.6], when the number of odd zeros $r_2 > 0$ or $D \neq 0$, $\widetilde{\mathcal{T}}_D$ and $\widetilde{\mathcal{P}}_D$ are abelian torsors over $\mathrm{Prym}(\widetilde{S}/\Sigma)$, with dimension $g(\widetilde{S}) - g = g - 1 + \frac{1}{2}r_2$. When $r_2 = 0$ and $D = 0$, $\widetilde{\mathcal{P}}_D$ and $\widetilde{\mathcal{T}}_D$ are torsors over $\mathrm{Nm}^{-1}(\mathcal{O}_\Sigma) \cup \mathrm{Nm}^{-1}(I)$, where Nm is the norm map of the covering $\tilde{\pi} : \tilde{S} \to \Sigma$, and $\mathcal{I}$ is the unique non-trivial line bundle that satisfies $\tilde{\pi}^*\mathcal{I} \cong \mathcal{O}_{\widetilde{S}}$. In addition, we define

$$\begin{aligned} \overline{\mathcal{T}}_D &= \{L \in \overline{\mathcal{T}} \mid p_{\mathrm{tf}}^\star L \in \widetilde{\mathcal{T}}_D\} \ ; \\ \overline{\mathcal{P}}_D &= \{L \in \overline{\mathcal{P}} \mid p_{\mathrm{tf}}^\star L \in \widetilde{\mathcal{P}}_D\}. \end{aligned} \tag{18}$$

Then the partial order on divisors defines a stratification of $\overline{\mathcal{T}}$ (resp. $\overline{\mathcal{P}}$) by $\cup_{D' \leq D} \overline{\mathcal{T}}_{D'}$ (resp. $\cup_{D' \leq D} \overline{\mathcal{P}}_{D'}$). The top strata are $\overline{\mathcal{T}}_{D=0}$ (resp. $\overline{\mathcal{P}}_{D=0}$), and these consist of the locally free sheaves. From the definition, $\mathcal{T} = \overline{\mathcal{T}}_{D=0}$ and $\mathcal{P} = \overline{\mathcal{P}}_{D=0}$.

Theorem 5.11. *(i) Suppose q contains at least one zero of odd order. For each stratum indexed by a σ-divisor D, if we let n_{ss} be the number of p such that $D|_p = \Lambda|_p$, then there are holomorphic fibre bundles*

$$\begin{aligned} (\mathbb{C}^*)^{k_1} \times \mathbb{C}^{k_2} &\longrightarrow \overline{\mathcal{T}}_D \xrightarrow{p_{\mathrm{tf}}^\star} \widetilde{\mathcal{T}}_D \ ; \\ (\mathbb{C}^*)^{k_1} \times \mathbb{C}^{k_2} &\longrightarrow \overline{\mathcal{P}}_D \xrightarrow{p_{\mathrm{tf}}^\star} \widetilde{\mathcal{P}}_D \end{aligned} \tag{19}$$

where $k_1 = r_1 - n_{ss}$, $k_2 = 2g - 2 - \frac{1}{2}\deg(D) - r_1 + n_{ss} - \frac{r_2}{2}$ and r_1, r_2 are the number of even and odd zeros.[1]

(ii) Suppose q is irreducible but all zeros are of even order. Then there exists an analytic space $\overline{\mathcal{T}}'_D$ and a double branched covering $p : \overline{\mathcal{T}}_D \to \overline{\mathcal{T}}'_D$, with $\overline{\mathcal{T}}'_D$ a holomorphic fibration

$$(\mathbb{C}^*)^{k_1} \times \mathbb{C}^{k_2} \longrightarrow \overline{\mathcal{T}}'_D \xrightarrow{p_{\mathrm{tf}}^\star} \widetilde{\mathcal{T}}_D.$$

In particular, $\dim(\overline{\mathcal{P}}_D) = \dim(\overline{\mathcal{T}}_D) = 3g - 3 - \frac{1}{2}\deg(D)$.

As explained in [Hor22a], via the BNR correspondence the stratification above translates into a stratification of the Hitchin fibre. Let $\chi_{\mathrm{BNR}} : \overline{\mathcal{T}} \xrightarrow{\sim} \mathcal{M}_q$ be the bijection in Theorem 2.3. Let D be a σ-divisor. Define $\mathcal{M}_{q,D} := \chi_{\mathrm{BNR}}(\overline{\mathcal{T}}_D)$. Then the stratification of $\overline{\mathcal{T}}$ induces a stratification on $\mathcal{M}_q = \bigcup_D \mathcal{M}_{q,D}$.

For each σ-divisor D, since $\sigma^* D = D$ and for any $x \in \mathrm{Fix}(\sigma)$, $D|_x = d_x x$, where $d_x \equiv 0 \mod 2$, we can write $D' := \frac{1}{2}\tilde{\pi}(D)$. Then D' is an effective divisor with $\operatorname{supp} D' \subset Z$. Moreover, for $x \in q^{-1}(0)$, $D'_x \leq \frac{1}{2}\lfloor \mathrm{ord}_x(q) \rfloor$. Therefore, $\mathcal{M}_q$ may be regarded as also being stratified by divisors D' defined over Σ.

5.6 The structure of the parabolic module projection

We now explain the relationship between the divisor D_v in Theorem 5.5 and the σ-divisor. Given $L \in \overline{\mathcal{P}}$, define

$$\begin{aligned} \mathscr{N}_L &:= \{(J, v) \in \widehat{\mathrm{PMod}}(\widetilde{S}) \mid \tau(J, v) = L\} \ ; \\ \mathscr{D}_L &:= \{D_v \mid (J, v) \in \mathscr{N}_L\} \end{aligned} \tag{20}$$

[1] J. Horn kindly pointed out to us that the formula in the paper [Hor22a, Theorem 6.2] needs to be modified by incorporating n_{ss}. The expressions for k_1 and k_2 are derived from [Hor22a, Proposition 5.12] and [Hor22a, Theorem 5.13]. Specifically, in [Hor22a, Proposition 5.12], it is stated that the local contribution of p is null when $D|_p = \Lambda|_p$, which leads to the expression $k_1 = r_1 - n_{ss}$.

That is, $\mathscr{N}_L = \tau^{-1}(L)$, and $\mathscr{D}_L$ is the collection of divisors D_v such that $J(-D_v) = p_{\mathrm{tf}}^{\star}(L)$. By Theorem 5.5, if $\tau(J, v) = \tau(J', v)$, then $J' = J(D_{v'} - D_v)$. By (16), as $L \in \overline{\mathcal{P}}$, we have $J, J' \in \mathrm{Prym}(\widetilde{S}/\Sigma)$, which implies $D_v = D_{v'}$. Therefore, for the cardinalities, we have $|\mathscr{N}_L| = |\mathscr{D}_L|$. Furthermore, we define $N_L := |\mathscr{N}_L| = |\mathscr{D}_L|$.

The divisor D_v satisfies the following symmetry proposition:

PROPOSITION 5.12. *Let D be a σ-divisor and $L \in \overline{\mathcal{P}}_D$. For any $D_v \in \mathscr{D}_L$, we have $D_v + \sigma^* D_v = D$.*

Proof. Let $\tau(J, v) = L$. Then by Theorem 5.5, we have $\widetilde{L} = J(-D_v)$, where $\widetilde{L} = p_{\mathrm{tf}}^{\star}(L)$. As $L \in \overline{\mathcal{P}}_D$ and $J \in \mathrm{Prym}(\widetilde{S}/\Sigma)$, we have $\widetilde{L} \otimes \sigma^* \widetilde{L} = \mathcal{O}(-D)$ and $J \otimes \sigma^* J = \mathcal{O}_{\widetilde{S}}$, which implies $D_v + \sigma^* D_v = D$. □

As a consequence, we have the following:

COROLLARY 5.13. *Suppose q has only zeros of odd order. Then for $L \in \overline{\mathcal{P}}_D$ and $D_v \in \mathscr{D}_L$, we have $\sigma^* D_v = D_v$ and $D_v = \frac{1}{2} D$. In addition, $\tau : \widehat{\mathrm{PMod}}(\widetilde{S}) \to \overline{\mathcal{P}}$ is a bijection.*

Proof. Since each zero has odd order, $\mathrm{supp}\,(D_v) \subset \mathrm{Fix}(\sigma)$, which implies $D_v = \sigma^* D_v$. By Proposition 5.12, we must have $D_v = \frac{1}{2} D$. □

There are relationships among the integers appearing in the construction of the parabolic module.

LEMMA 5.14. *Let $D = \sum_{i=1}^{r_1} d_i(\tilde{p}_i^+ + \tilde{p}_i^-) + \sum_{i=1}^{r_2} d_i' \tilde{p}_i'$ be a σ-divisor, and let $L \in \overline{\mathcal{P}}_D$. Then we have*

(i) $\ell_{p_i} = d_i$ *and* $\ell_{p_i'} = d_i'/2$;

(ii) $a_{\tilde{p}_i^+} = a_{\tilde{p}_i^-} = (m_i/2) - d_i$ *and* $a_{\tilde{p}_i'} = n_i - 1 - d_i'$.

Proof. Since $L \in \overline{\mathcal{P}}_D$, we have $\dim T(p^* L_{p_i}) = d_i$ and $\dim T(p^* L_{p_i'}) = d_i'/2$. The claim then follows from Proposition 9.2.1. □

PROPOSITION 5.15. *Let $D = \sum_{i=1}^{r_1} d_i(\tilde{p}_i^+ + \tilde{p}_i^-) + \sum_{i=1}^{r_2} d_i' \tilde{p}_i'$ be a σ-divisor, and let $L \in \overline{\mathcal{P}}_D$. Then $N_L = \prod_{i=1}^{r_1}(d_i + 1)$. The number N_L depends only on the σ-divisor D.*

Proof. By Lemma 5.14, $V(L)$ can be rewritten as

$$V(L) = \{(c_1^{\pm}, \ldots, c_{r_1}^{\pm}, c_1' = l_{p_1'}, \ldots, c_{r_2}' = l_{p_{r_2}'}) \mid c_i^+ + c_i^- = d_i, c_i^{\pm} \in \mathbb{Z}_{\geq 0}\}.$$

□

If we define n_L to be the number of $D_v \in \mathscr{D}_L$ such that $\sigma^* D_v \neq D_v$, then we have the following:

PROPOSITION 5.16

(i) n_L *is even;*

(ii) if $L \in \overline{\mathcal{P}}_D$ *with*

$$D = \sum_{i=1}^{r_1} d_i(\tilde{p}_i^+ + \tilde{p}_i^-) + \sum_{i=1}^{r_2} d_i' \tilde{p}_i'$$

and if there exists $i_0 \in \{1, \ldots, r_1\}$ such that d_{i_0} is not even, then $n_L = N_L$; otherwise, $n_L = N_L - 1$.

Proof. To prove (i), note that if $\sigma^* D_v \neq D_v$, then $\sigma^*(\sigma^* D_v) \neq \sigma^* D_v$, which means that n_L is even. For (ii), by Proposition 5.15, $D_v = \sigma^* D_v$ for $D_v \in \mathscr{D}_L$ if and only if $c_i^+ = c_i^- = d_i/2$. Therefore, $n_L \neq N_L$ if and only if all d_i are even, which implies (ii). □

We should note that the integer n_L depends only on the Higgs divisor D, and in the rest of this article, we define $n_D := \frac{n_L}{2}$.

6. Irreducible singular fibres and the Mochizuki map

In this section, we provide a reinterpretation of the limiting configuration construction of a Higgs bundle on an irreducible fibre, as introduced by Mochizuki in [Moc16] (see also [Hor22a]). We also investigate the relationship between limiting configurations and the stratification.

6.1 Abelianization of a Higgs bundle

Let q be a fixed irreducible quadratic differential with spectral curve S and with normalisation $p : \widetilde{S} \to S$. We define $\widetilde{K} := \widetilde{\pi}^* K$ (but note that $\widetilde{K} \neq K_{\widetilde{S}}$) and $\widetilde{q} := \widetilde{\pi}^* q \in H^0(\widetilde{K}^2)$, where $\widetilde{\pi}$ is as in (10). Choose a square root $\omega \in H^0(\widetilde{K})$ such that $\widetilde{q} = -\omega \otimes \omega$ (that is, $\omega = p^*\lambda$). Let $\Lambda := \mathrm{Div}(\omega)$ and $\widetilde{Z} := \mathrm{supp}(\Lambda)$. We can then write

$$\Lambda = \sum_{i=1}^{r_1} \frac{m_i}{2}(\tilde{p}_i^+ + \tilde{p}_i^-) + \sum_{j=1}^{r_2} n_j \tilde{p}_j'. \tag{21}$$

If $\sigma : \widetilde{S} \to \widetilde{S}$ denotes the involution, then $\sigma^*\omega = -\omega$.

Let $(\mathcal{E}, \varphi)$ be a Higgs bundle on Σ with $\det\varphi = q$. Consider the pull-back $(\widetilde{\mathcal{E}}, \tilde{\varphi}) := (\widetilde{\pi}^*\mathcal{E}, \widetilde{\pi}^*\varphi)$ to $\widetilde{S}$. We have $\tilde{\varphi} \in H^0(\mathrm{End}(\widetilde{\mathcal{E}}) \otimes \widetilde{K})$ and $\widetilde{q} = \det(\tilde{\varphi})$. Since $\widetilde{q} = -\omega \otimes \omega$, $\pm\omega$ are well-defined eigenvalues of $\tilde{\varphi}$ over $\widetilde{S}$. Let $\tilde{\lambda}$ be the canonical section of the pull-back of $\widetilde{K}$ to the total space $\mathrm{Tot}(\widetilde{K})$. The spectral curve for $(\widetilde{\mathcal{E}}, \tilde{\varphi})$ is defined by the equation

$$\widetilde{S}' := \{\tilde{\lambda}^2 - \tilde{q} = 0\}.$$

The set $\widetilde{S}' = \mathrm{Im}(\omega) \cup \mathrm{Im}(-\omega) \subset \mathrm{Tot}(\widetilde{K})$ decomposes into two irreducible pieces:

Having fixed a choice of ω, the eigenvalues of $\tilde{\varphi}$ are globally well-defined, and we can define the line bundle $\widetilde{L}_+ \subset \widetilde{\mathcal{E}}$ as $\widetilde{L}_{\pm} := \ker(\tilde{\varphi} - \omega)$. Since $\sigma^*\omega = -\omega$, $\widetilde{L}_- = \sigma^*\widetilde{L}_+ = \ker(\tilde{\varphi} + \omega)$, and there is an isomorphism $\widetilde{\mathcal{E}}|_{\widetilde{S}\setminus\widetilde{Z}} \cong \widetilde{L}_+ \oplus \widetilde{L}_-|_{\widetilde{S}\setminus\widetilde{Z}}$.

There is also a local description of $(\widetilde{\mathcal{E}}, \tilde{\varphi})$.

Lemma 6.1. *Let $x \in \widetilde{Z}$ and write $\Lambda|_x = m_x x$. Let U be a holomorphic coordinate neighborhood of x. Then there exists a frame $\mathfrak{e} \in H^0(U, \widetilde{K})$ such that, under a suitable trivialization of $\mathcal{E}|_U \cong U \times \mathbb{C}^2$, we can write*

$$\tilde{\varphi} = z^{d_x} \begin{pmatrix} 0 & 1 \\ z^{2m_x - 2d_x} & 0 \end{pmatrix} \otimes \mathfrak{e}. \tag{22}$$

Moreover, if we define $D := \sum_{x \in \widetilde{Z}} d_x x$, then D is a σ-divisor.

Lemma 6.2. *For the $\widetilde{L}_{\pm}$ defined above, we have $\widetilde{L}_+ \otimes \widetilde{L}_- = \mathcal{O}_{\widetilde{S}}(D - \Lambda)$. Moreover, if we denote $\widetilde{L}_0 := \widetilde{L}_+(\Lambda - D)$ and $\widetilde{L}_1 := \sigma^*\widetilde{L}_0$, then $\widetilde{L}_+ = \widetilde{\mathcal{E}} \cap \widetilde{L}_0$, $\widetilde{L}_- = \widetilde{\mathcal{E}} \cap \widetilde{L}_1$, and we have the exact sequences*

$$0 \longrightarrow \widetilde{L}_+ \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \widetilde{L}_1 \longrightarrow 0\ ;$$
$$0 \longrightarrow \widetilde{L}_- \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \widetilde{L}_0 \longrightarrow 0.$$

Proof. The inclusion of $\widetilde{L}_\pm \to \widetilde{\mathcal{E}}$ defines an exact sequence:

$$0 \longrightarrow \widetilde{L}_+ \oplus \widetilde{L}_- \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \mathcal{T} \longrightarrow 0$$

where $\mathcal{T}$ is a torsion sheaf with $\operatorname{supp}\mathcal{T} \subset \widetilde{Z}$. From the local description in (22), in the same trivialization, $\widetilde{L}_\pm$ are spanned by the bases $s_\pm = \binom{1}{\pm z^{m_x - d_x}}$. Therefore, as $\det(\mathcal{E}) = \mathcal{O}_\Sigma$, we obtain $\widetilde{L}_+ \otimes \widetilde{L}_- = \mathcal{O}_{\widetilde{S}}(D - \Lambda)$. Since s_+, s_- are linear independent away from z, $\widetilde{\mathcal{E}}/\widetilde{L}_+$ is locally generated by the section $z^{d_x - m_x} s_-$. Therefore, $\widetilde{\mathcal{E}}/\widetilde{L}_+ \cong \widetilde{L}_-(\Lambda - D) = \widetilde{L}_1$. Using the involution, we obtain the other exact sequence. ☐

Therefore, if $\widetilde{L} \otimes \sigma^* \widetilde{L} = \mathcal{O}_{\widetilde{S}}(D - \Lambda)$, we have $\widetilde{L}_0 = \widetilde{L}(\Lambda - D) \in \widetilde{\mathcal{T}}_D$. In summary, the construction above leads us to consider the composition of the following maps given by the composition

$$\delta : \mathcal{M}_q \to \widetilde{\mathcal{T}}_D, \ \ (\mathcal{E}, \varphi) \mapsto \widetilde{L}_+ \mapsto \widetilde{L}_+(\Lambda - D)$$

where the first map is obtained by taking the kernel of $(\tilde{\pi}^* \varphi - \omega)|_{\tilde{\pi}^* \mathcal{E}}$.

This procedure is directly related to the torsion-free pull-back. Recall that $\chi_{\mathrm{BNR}} : \overline{\mathcal{T}} \to \mathcal{M}_q$ is the BNR correspondence map and that $p^\star_{\mathrm{tf}} : \overline{\operatorname{Pic}}(S) \to \operatorname{Pic}(\widetilde{S})$ is the torsion-free pull-back. Then we have

Proposition 6.3. *$\delta \circ \chi_{\mathrm{BNR}} = p^\star_{\mathrm{tf}}$. In particular, if $J \in \overline{\mathcal{T}}_D$, then $\delta \circ \chi_{\mathrm{BNR}}(J) \in \widetilde{\mathcal{T}}_D$.*

Proof. Let $J \in \overline{\mathcal{T}}$, and write $(\mathcal{E}, \varphi) = \chi_{\mathrm{BNR}}(J)$ and $(\widetilde{\mathcal{E}}, \tilde{\varphi}) := \tilde{\pi}^*(\mathcal{E}, \varphi)$. Recall the BNR exact sequence on S (see (3)). As p^* is right-hand-side exact, we obtain

$$\widetilde{\mathcal{E}} \xrightarrow{\tilde{\varphi} - \tilde{\lambda}} \widetilde{\mathcal{E}} \otimes \tilde{K} \longrightarrow p^* J \otimes \tilde{K} \longrightarrow 0.$$

Since the spectral curve is $\widetilde{S}' = \operatorname{Im}(\omega) \cup \operatorname{Im}(-\omega)$, we can consider the restriction to the component $\operatorname{Im}(\omega)$ and write $\tilde{\lambda} = \omega$, $\widetilde{L}_\pm := \ker(\tilde{\varphi} \mp \omega)$. We obtain the exact sequence

$$0 \longrightarrow \widetilde{L}_+ \longrightarrow \widetilde{\mathcal{E}} \xrightarrow{\tilde{\varphi} - \omega} \widetilde{\mathcal{E}} \otimes \tilde{K} \longrightarrow p^* J \otimes \tilde{K} \longrightarrow 0$$

which breaks into short exact sequences

$$0 \longrightarrow \widetilde{L}_+ \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \operatorname{Im}(\tilde{\varphi} - \omega) \longrightarrow 0$$
$$0 \longrightarrow \operatorname{Im}(\tilde{\varphi} - \omega) \longrightarrow \widetilde{\mathcal{E}} \otimes \tilde{K} \longrightarrow p^* J \otimes \tilde{K} \longrightarrow 0.$$

Using the local trivialization in Lemma 6.1, $\operatorname{Im}(\tilde{\varphi} - \omega)$ is locally spanned by $\binom{z^{d_x}}{-z^{m_x}}\mathfrak{e}$. From Lemma 6.2, if we write $\widetilde{L}_0 := \widetilde{L}_+(\Lambda - D)$ and $\widetilde{L}_1 := \sigma^* \widetilde{L}_0$, then

$$\delta \circ \chi_{\mathrm{BNR}}(J) = \widetilde{L}_+(\Lambda - D).$$

Moreover, there is an isomorphism $\operatorname{Im}(\tilde{\varphi} - \omega) \cong \widetilde{L}_1$. Letting $\widetilde{L}_1'$ be the saturation of $\widetilde{L}_1$, we obtain the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{L}_1 & \longrightarrow & \widetilde{\mathcal{E}} \otimes \tilde{K} & \longrightarrow & p^* J \otimes \tilde{K} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \\
0 & \longrightarrow & \widetilde{L}_1' & \longrightarrow & \widetilde{\mathcal{E}} \otimes \tilde{K} & \longrightarrow & p^\star_{\mathrm{tf}} J \otimes \tilde{K} & \longrightarrow & 0
\end{array}$$

where $i:\widetilde{L}_1 \to \widetilde{L}_1'$ is the natural inclusion. Moreover, in the same trivialization, $\widetilde{L}_1'$ is spanned by the section $\left(\begin{smallmatrix}1\\ -z^{m_x-d_x}\end{smallmatrix}\right)\mathfrak{e}$. Therefore, $\widetilde{L}_1' \cong \widetilde{L}_- \otimes \widetilde{K}$, and from Lemma 6.2, $p_{\mathrm{tf}}^{\star}J = \delta \circ \chi_{\mathrm{BNR}}(J)$. □

If $(\mathcal{E},\varphi)$ is a Higgs bundle with $(\mathcal{E},\varphi) = \chi_{\mathrm{BNR}}(L)$ and $\widetilde{L}_0 = \delta \circ \chi_{\mathrm{BNR}}(L)$, then by Proposition 6.3, $\widetilde{L}_0 = p_{\mathrm{tf}}^{\star}(L)$. We define a Higgs bundle $(\widetilde{\mathcal{E}}_0, \tilde{\varphi}_0)$ as follows:

$$\widetilde{\mathcal{E}}_0 = \widetilde{L}_0 \oplus \sigma^* \widetilde{L}_0,\ \tilde{\varphi}_0 = \begin{pmatrix}\omega & 0\\ 0 & -\omega\end{pmatrix}.$$

Moreover, $\widetilde{\mathcal{E}}$ is an $\mathcal{O}_{\widetilde{S}}$ submodule of $\widetilde{\mathcal{E}}_0$, with a natural inclusion $\iota:\widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}_0$ satisfying the following:

(i) the induced morphism $\widetilde{\mathcal{E}} \to \widetilde{L}_0,\ \widetilde{\mathcal{E}} \to \sigma^*\widetilde{L}_0$ is surjective;

(ii) the restriction of $\iota|_{\widetilde{S}\setminus\widetilde{Z}}$ is an isomorphism;

(iii) $\tilde{\varphi}_0 \circ \iota = \iota \circ \tilde{\varphi}$.

Following [Moc16, Sec. 4.1], we call $(\widetilde{\mathcal{E}}_0, \tilde{\varphi}_0)$ the *abelianization of the Higgs bundle* $(\mathcal{E},\varphi)$.

6.2 The construction of the algebraic Mochizuki map

In this subsection, we define the algebraic Mochizuki map, as introduced in [Moc16]. Recall that for any divisor $D = \sum_{x\in Z} d_x x$, there is a canonical weight function

$$\chi_D(x) := \begin{cases} d_x & x \in \operatorname{supp} D\\ 0 & x \notin \operatorname{supp} D.\end{cases}$$

We also have the stratification $\overline{\mathcal{T}} = \cup_D \overline{\mathcal{T}}_D$ for σ divisors D. Let $\mathscr{F}(\widetilde{S})$ be the space of all degree-zero filtered-line bundles over $\widetilde{S}$. The *algebraic Mochizuki map* Θ^{Moc} is defined as

$$\Theta^{\mathrm{Moc}}:\overline{\mathcal{T}} \longrightarrow \mathscr{F}(\widetilde{S})\ \ L \mapsto \mathcal{F}_*(p_{\mathrm{tf}}^{\star}(L), \tfrac{1}{2}\chi_{D-\Lambda}).$$

Example 6.4. When q has only simple zeros, this construction generalises that of [MSWW16] (see also [Fre18]). In the case of a quadratic differential with simple zeros, the spectral curve S is smooth, and every torsion-free sheaf is locally free, so $\mathcal{T} = \overline{\mathcal{T}}$. If $Z = \{p_1,\dots,p_{4g-4}\}$ are the branch points of S and if $\Lambda = \sum_{i=1}^{4g-4} p_i$, then the weight function $\frac{1}{2}\chi_{-\Lambda}$ assigns a weight of $-\frac{1}{2}$ to each p_i. For $L \in \mathcal{T}$, $\Theta^{\mathrm{Moc}}(L) = \mathcal{F}_*(L, \frac{1}{2}\chi_{-\Lambda})$.

Below are some additional properties of Θ^{Moc}:

Proposition 6.5. *$\Theta^{\mathrm{Moc}}|_{\overline{\mathcal{T}}_D}$ is a continuous map.*

Proof. This follows directly from the definition of Θ^{Moc} and Theorem 3.3. □

From Theorem 5.11, we know that for a σ-divisor D, the preimage of the map $p_{\mathrm{tf}}^{\star}:\overline{\mathcal{T}}_D \to \widetilde{\mathcal{T}}_D$ has dimension $2g-2-\frac{1}{2}\deg(D) - r_2/2$, where r_2 is the number of odd zeros of q. Even for the top stratum $D=0$, $p_{\mathrm{tf}}^{\star}$ is not injective if the spectral curve is not smooth. Indeed, if $L_1, L_2 \in \overline{\mathcal{T}}_D$ with $p_{\mathrm{tf}}^{\star}(L_1) = p_{\mathrm{tf}}^{\star}(L_2)$, then based on the construction, we have $\Theta^{\mathrm{Moc}}(L_1) = \Theta^{\mathrm{Moc}}(L_2)$. In summary, we have the following result:

Proposition 6.6. *If $q \in H^0(K^2)$ is irreducible, then Θ^{Moc} is injective if and only if q has simple zeros.*

6.3 Convergence of subsequences

Fix a locally free $L_0 \in \mathcal{T}$. Using the isomorphism $\psi_{L_0} : \overline{\mathcal{T}} \to \overline{\mathcal{P}}$ defined by $\psi_{L_0}(L) = LL_0^{-1}$, we can extend the Mochizuki map Θ^{Moc} to $\overline{\mathcal{P}}$. For $J \in \overline{\mathcal{P}}_D$, we write $\widetilde{J} := p_{\mathrm{tf}}^{\star}(J)$ and choose the weight function $\frac{1}{2}\chi_D$. We then define:

$$\Theta_0^{\mathrm{Moc}} : \overline{\mathcal{P}}_D \longrightarrow \mathscr{F}(\widetilde{S}) \ \ J \mapsto \mathcal{F}_*(\widetilde{J}, \tfrac{1}{2}\chi_D).$$

Proposition 6.7. *The map Θ_0^{Moc} satisfies the following properties:*

(i) If $J \in \overline{\mathcal{P}}$ and $L := L_0 J$, then

$$\Theta_0^{\mathrm{Moc}}(J) = \Theta^{\mathrm{Moc}}(L) \otimes \Theta^{\mathrm{Moc}}(L_0)^{-1}$$

where $\otimes$ is the tensor product for filtered line bundles (7).

(ii) If $L = \tau(I, v)$, with $(I, v) \in \widehat{\mathrm{PMod}}(\widetilde{S})$ and with $L \in \overline{\mathcal{P}}_D$, then

$$\Theta_0^{\mathrm{Moc}} \circ \tau(I, v) = \mathcal{F}_*(I(-D_v), \tfrac{1}{2}\chi_{D_v + \sigma^* D_v})$$

where D_v is the corresponding divisor defined in Theorem 5.5.

(iii) If $\sigma^* D_v = D_v$, then $\Theta_0^{\mathrm{Moc}} \circ \tau(I, v) = \mathcal{F}_*(I, 0)$, where 0 means that all parabolic weights are zero.

Proof. As L_0 is locally free, we have $p_{\mathrm{tf}}^{\star} J = (p^* L_0)^{-1} \otimes p_{\mathrm{tf}}^{\star} L$. By definition,

$$\Theta_0^{\mathrm{Moc}}(J) = \mathcal{F}_*(p_{\mathrm{tf}}^{\star} J, \tfrac{1}{2}\chi_D) \ \ \Theta^{\mathrm{Moc}}(L) = \mathcal{F}_*(p_{\mathrm{tf}}^{\star} L, \tfrac{1}{2}\chi_{D-\Lambda}) \ \ \textit{and} \ \ \Theta^{\mathrm{Moc}}(L_0) = \mathcal{F}_*(p_{\mathrm{tf}}^{\star} L_0, \tfrac{1}{2}\chi_{-\Lambda})$$

which implies (i). For (ii), by Theorem 5.5, $p_{\mathrm{tf}}^{\star} L = I(-D_v)$, and from Proposition 5.12, we have $D = D_v + \sigma^* D_v$, which implies (ii). When $\sigma^* D_v = D_v$, we compute

$$\mathcal{F}_*(I(-D_v) \ \ \tfrac{1}{2}\chi_{D_v + \sigma^* D_v}) = \mathcal{F}_*(I(-D_v) \ \ \chi_{D_v}) = \mathcal{F}_*(I, 0)$$

which implies (iii). □

We now give a criterion for the continuity of the map Θ^{Moc}. By Proposition 6.7, it is sufficient to study the map Θ_0^{Moc}. Recall that for $L \in \overline{\mathcal{P}}$, we have

$$\mathscr{N}_L := \{(J, v) \in \widehat{\mathrm{PMod}}(\widetilde{S}) \mid \tau(J, v) = L\}, \quad \mathscr{D}_L := \{D_v \mid (J, v) \in \mathscr{N}_L\}$$

and that the number n_L is defined to be the number of divisors $D_v \in \mathscr{D}_L$ such that $\sigma^* D_v \neq D_v$.

Proposition 6.8. *Let D be a σ-divisor, with $L \in \overline{\mathcal{P}}_D$, and assume that Θ_0^{Moc} is continuous at L. Then, for $(J, v) \in \mathscr{N}_L$ and $D_v \in \mathscr{D}_L$, we have $\sigma^* D_v = D_v$, i.e., $n_L = 0$.*

Proof. As the top stratum $\mathcal{P}$ is dense in $\overline{\mathcal{P}}$, there exists a family $L_i \in \mathcal{P}$ such that $\lim_{i\to\infty} L_i = L$. Let $(J_i, v_i) \in \widehat{\mathrm{PMod}}(\widetilde{S})$ be such that $\tau(J_i, v_i) = L_i$. Then, after passing to subsequences, $\lim_{i\to\infty}(J_i, v_i) = (J_\infty, v_\infty)$, and $\tau(J_\infty, v_\infty) = L$. As L_i is locally free, we have $D_{v_i} = 0$. Moreover, by Theorem 5.5, we have $p_{\mathrm{tf}}^{\star} L = J_\infty(-D_{v_\infty})$, and from Proposition 5.12, we have $D = D_{v_\infty} + \sigma^* D_{v_\infty}$. By Proposition 6.3.1, we have

$$\Theta_0^{\mathrm{Moc}}(L_i) = \Theta_0^{\mathrm{Moc}} \circ \tau(J_i, v_i) = \mathcal{F}_*(J_i, 0)$$

and we compute

$$\lim_{i\to\infty} \Theta_0^{\mathrm{Moc}}(L_i) = \mathcal{F}_*(J_\infty, 0) = \mathcal{F}_*(J_\infty(-D_{v_\infty}), \chi_{D_{v_\infty}}).$$

Moreover, by Proposition 6.3.1, we have

$$\Theta_0^{\mathrm{Moc}}(L) = \mathcal{F}_*(J_\infty(-D_{v_\infty}), \frac{1}{2}(\chi_{D_{v_\infty}} + \chi_{\sigma^* D_{v_\infty}})).$$

Since Θ_0^{Moc} is continuous on L, we have $\lim_{i\to\infty} \Theta_0^{\mathrm{Moc}}(L_i) = \Theta^{\mathrm{Moc}}(L)$, which implies that $\chi_{D_{v_\infty}} = \chi_{\sigma^* D_{v_\infty}}$. □

By Proposition 5.16, $n_L > 0$ if and only if q has at least one zero of even order. Hence, the following is immediate:

Corollary 6.9. *Suppose q is irreducible and has a zero of even order. Then Θ_0^{Moc} is not continuous.*

By contrast, we have the following:

Proposition 6.10. *If q is irreducible with all zeros of odd order, then Θ_0^{Moc} is continuous.*

Proof. Since all zeros of q are odd, for any $L \in \overline{\mathcal{P}}$ we have $n_L = 0$. Let $L_\infty \in \overline{\mathcal{P}}$ be fixed, and let $L_i \in \overline{\mathcal{P}}$ be any sequence such that $\lim_{i\to\infty} L_i = L_\infty$. Since $\tau : \widehat{\mathrm{PMod}}(\widetilde{S}) \to \overline{\mathcal{P}}$ is bijective, we take $(J_i, v_i) \in \widehat{\mathrm{PMod}}(\widetilde{S})$ with $\tau(J_i, v_i) = L_i$. Moreover, we assume $\lim_{i\to\infty}(J_i, v_i) = (J_\infty, v_\infty)$ with $\tau(J_\infty, v_\infty) = L_\infty$. Since q contains only odd-order zeros, it follows that $\operatorname{supp} D_v \subset \mathrm{Fix}(\sigma)$. By Proposition 6.3.1, we have $\Theta_0^{\mathrm{Moc}}(L_i) = \mathcal{F}_*(J_i, 0)$. Therefore, we have:

$$\lim_{i\to\infty} \Theta_0^{\mathrm{Moc}}(L_i) = \lim_{i\to\infty} \mathcal{F}_*(J_i, 0) = \mathcal{F}_*(J_\infty, 0) = \Theta_0^{\mathrm{Moc}}(L_\infty).$$

This concludes the proof. □

Theorem 6.11. *Suppose q is irreducible. For the map $\Theta^{\mathrm{Moc}} : \mathcal{M}_q \to \mathscr{F}(\widetilde{S})$, we have:*

(i) Θ^{Moc} is injective if and only if q has only simple zeros;

(ii) if q has only zeros of odd order, Θ^{Moc} is continuous;

(iii) if q contains a zero of even order, Θ^{Moc} is not continuous.

Proof. (i) follows from Proposition 6.6. (ii) follows from Proposition 6.10. (iii) follows from Corollary 6.9. □

Proposition 6.12. *Suppose $n_L > 0$. Then for $k = 1, \ldots, n_L$, there exist sequences $L_i^k \in \mathcal{P}$ with $\lim_{i\to\infty} L_i^k = L$ such that if we denote $\mathcal{F}_*^k := \lim_{i\to\infty} \Theta_0^{\mathrm{Moc}}(L_i^k)$ and $\mathcal{F}_*^0 := \Theta_0^{\mathrm{Moc}}(L)$, then $\mathcal{F}_*^{k_1} \neq \mathcal{F}_*^{k_2}$ for $k_1 \neq k_2$. Moreover, there exist $\{D_1, \ldots, D_{n_L}\} \subset \mathscr{D}_L$ such that $\mathcal{F}_*^k = \mathcal{F}_*(p_{\mathrm{tf}}^\star L, \chi_{D_k})$.*

Proof. By the definition of n_L, we can find (J^k, v^k) with $\tau(J^k, v^k) = L$. If we define $D_k := D_{v^k}$, then $\sigma^* D_k \neq D_k$. Moreover, by Theorem 5.5, we have $p_{\mathrm{tf}}^\star L = J^k(-D_k)$. As $\tau^{-1}(\mathcal{P})$ is dense in $\widehat{\mathrm{PMod}}(\widetilde{S})$, for each (J^k, v^k), we can find a sequence $(J_i^k, v_i^k) \in \tau^{-1}(\mathcal{P})$ such that $\lim_{i\to\infty}(J_i^k, v_i^k) = (J^k, v^k)$, and we define $L_i^k := \tau(J_i^k, v_i^k)$. Since L_i^k is locally free, $D_{v_i^k} = 0$, and thus $\Theta_0^{\mathrm{Moc}}(L_i^k) = \mathcal{F}_*(J_i^k, 0)$. We compute

$$\lim_{i\to\infty} \Theta_0^{\mathrm{Moc}}(L_i^k) = \mathcal{F}_*(J^k, 0) = \mathcal{F}_*(p_{\mathrm{tf}}^\star L, \chi_{D_k})$$

and $\Theta_0^{\mathrm{Moc}}(L) = \mathcal{F}_*(p_{\mathrm{tf}}^\star L, \frac{1}{2}\chi_D)$. Based on our assumptions, we have $D_{k_1} \neq D_{k_2}$ for $k_1 \neq k_2$ and $\sigma^* D_k \neq D_k$, which implies that $\chi_{D_{k_1}} \neq \chi_{D_{k_2}}$ for $k_1 \neq k_2$ and $\chi_{D_k} \neq \frac{1}{2}\chi_D$. □

We now present a computation for the case of a simple nodal curve:

Example 6.13. Let q be a quadratic differential with $2g-4$ simple zeros, and let x be an even zero of q of order two. Then S has a singular point, which we also denote by x. Let $p:\widetilde{S}\to S$ be the normalisation map and let $p^{-1}(x)=\{x_1,x_2\}$. Consider the σ-divisor $D=x_1+x_2$, and let $L\in\overline{\mathcal{P}}_D$. Then $n_L=2$, and we can write $\mathscr{N}_L=(J_1,v_1),(J_2,v_2)$, where $D_{v_1}=x_1$ and $D_{v_2}=x_2$. Moreover, we have $p_{\mathrm{tf}}^{\star}L=J_1\otimes\mathcal{O}(-x_1)=J_2\otimes\mathcal{O}(-x_2)$. Let (α,β) denote the parabolic weight that is equal to α at x_1, β at x_2, and $\frac{1}{2}$ at all other zeros. Then the filtered bundles obtained in Proposition 6.12 are

$$\mathcal{F}_*(p_{\mathrm{tf}}^{\star}L,(1,0))\ \ \mathcal{F}_*(p_{\mathrm{tf}}^{\star}L,(0,1))\ \ \mathcal{F}_*(p_{\mathrm{tf}}^{\star}L,(\tfrac{1}{2},\tfrac{1}{2})).$$

6.4 Mochizuki's convergence theorem for irreducible fibres

In this subsection, we recall Mochizuki's construction of the limiting configuration metric [Moc16, Section 4.2.1, 4.3.2] and the convergence theorem.

6.4.1 *Limiting configuration metric.* Let q be an irreducible quadratic differential, and let $(\mathcal{E},\varphi)\in\mathcal{M}_q$ be a Higgs bundle with $(\mathcal{E},\varphi)=\chi_{\mathrm{BNR}}(L)$. We write $\widetilde{L}_0=p_{\mathrm{tf}}^{\star}L$ and $(\widetilde{\mathcal{E}},\tilde{\varphi}):=p^*(\mathcal{E},\varphi)$. Then the abelianization of $(\mathcal{E},\varphi)$, which is a Higgs bundle over $\widetilde{S}$, can be written as $\widetilde{\mathcal{E}}_0=\widetilde{L}_0\oplus\sigma^*\widetilde{L}_0$, $\tilde{\varphi}_0=\mathrm{diag}(\omega,\,-\omega)$. The natural inclusion $\iota:(\widetilde{\mathcal{E}},\tilde{\varphi})\to(\widetilde{\mathcal{E}}_0,\tilde{\varphi}_0)$ is an isomorphism over $\widetilde{S}\setminus\widetilde{Z}$. Moreover, we let D be the σ-divisor of $(\mathcal{E},\varphi)$.

From the construction of $\Theta^{\mathrm{Moc}}(L)$ and Proposition 6.12, we have n_L different divisors D_k for $k=1,\cdots,n_L$ with $\sigma^*D_k\neq D_k$ and $D_k+\sigma^*D_k=D$. Moreover, we can find n_L+1 different filtered bundles with deg 0. Define

$$\mathcal{F}_{*,0}:=\Theta^{\mathrm{Moc}}(L)=\mathcal{F}_*(\widetilde{L}_0,\chi_{\frac{1}{2}(D-\Lambda)}),\ \mathcal{F}_{*,k}:=\mathcal{F}_*(\widetilde{L}_0,\chi_{(D_k-\frac{1}{2}\Lambda)})$$

which are all degree-zero filtered bundles with different level of filtrations.

Now we will introduce the construction in [Moc16, Section 4.2.1, 4.3.2]. For $k=0,\cdots,n_L$, we define $\tilde{h}_k$ to be the harmonic metric for the filtered bundle $\mathcal{F}_{*,k}$; this is well-defined up to a positive multiplicative constant. To fix this constant, assume that $\sigma^*\tilde{h}_k\otimes\tilde{h}_k=1$. This gives a unique choice of $\tilde{h}_k$. We then define the metric $\widetilde{H}_k=\mathrm{diag}(\tilde{h}_k,\,\sigma^*\tilde{h}_k)$ on $\widetilde{\mathcal{E}}_0$, with $\det(\widetilde{H}_k)=1$. For the resulting harmonic bundle $(\widetilde{\mathcal{E}}_0,\varphi_0,\widetilde{H}_k)$, we define $\tilde{\nabla}_k$ to be the unitary connection determined by $\widetilde{H}_k$. Since $\widetilde{H}_k$ is diagonal over $\widetilde{S}\setminus\widetilde{Z}$, it follows that $F_{\tilde{\nabla}_k}=0$, and we have $[\varphi_0,\varphi_0^{\dagger_{\widetilde{H}_k}}]=0$. Furthermore, as ι is an isomorphism on $\widetilde{S}\setminus\widetilde{Z}$, the metric $\widetilde{H}_k$ also defines a metric on $(\widetilde{\mathcal{E}},\tilde{\varphi})$ over $\widetilde{S}\setminus\widetilde{Z}$.

For any $\tilde{x}\in\widetilde{S}\setminus\widetilde{Z}$ with $x:=p(\tilde{x})$, we have the isomorphisms

$$(\widetilde{\mathcal{E}}_0,\tilde{\varphi}_0)|_{\sigma(\tilde{x})}\cong(\widetilde{\mathcal{E}}_0,\tilde{\varphi}_0)|_{\tilde{x}}\cong(\widetilde{\mathcal{E}},\tilde{\varphi})|_{\tilde{x}}\cong(\mathcal{E},\varphi)|_x.$$

Therefore, $\widetilde{H}_k$ induces a metric H_k^{Lim} on $\Sigma\setminus Z$, and we may consider H_k^{Lim} as the push-forward of $\tilde{h}_k$. In [Hor22b, Theorem 5.2], the push-forward metric of $\Theta^{\mathrm{Moc}}(L)$ is explicitly written in local coordinates.

Recall the notation from Section 2.4. Let E be a trivial, smooth, rank 2 vector bundle over a Riemann surface Σ, and let H_0 be a background Hermitian metric on E. Over $\Sigma\setminus Z$, we write ∇_k^{Lim} for the Chern connection defined by H_k^{Lim}, which is unitary with respect to H_0, and let $\phi_k^{\mathrm{Lim}}=\varphi_k^{\mathrm{Lim}}+\varphi_k^{\dagger_{\mathrm{Lim}}}$ be the corresponding Higgs field in the unitary gauge. They both satisfy the decoupled Hitchin equations over $\Sigma \setminus Z$. Thus, from any Higgs bundle $(\mathcal{E}, \varphi)$, we obtain $n_L + 1$ limiting configurations:

$$(\nabla_k^{\mathrm{Lim}}, \phi_k^{\mathrm{Lim}} = \varphi + \varphi_k^{\dagger_{\mathrm{Lim}}}) \in \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}.$$

The flat connection, which is defined over $\Sigma \setminus Z$, may be understood by using the nonabelian Hodge correspondence for filtered vector bundles [Sim90]. Given filtered line bundles $\mathcal{F}_{*,k}$, define filtered vector bundles $\widetilde{\mathcal{E}}_{*,k} := \mathcal{F}_{*,k} \oplus \sigma^* \mathcal{F}_{*,k}$, which can be explicitly written as

$$\begin{aligned} &\widetilde{\mathcal{E}}_{*,0} := \mathcal{F}_*(\widetilde{L}_0, \chi_{\frac{1}{2}(D-\Lambda)}) \oplus \mathcal{F}_*(\sigma^* \widetilde{L}_0, \chi_{\frac{1}{2}(D-\Lambda)}) \ ; \\ &\widetilde{\mathcal{E}}_{*,k} := \mathcal{F}_*(\widetilde{L}_0, \chi_{D_k - \frac{1}{2}\Lambda}) \oplus \mathcal{F}_*(\sigma^* \widetilde{L}_0, \chi_{\sigma^* D_k - \frac{1}{2}\Lambda}) \ \ k \neq 0. \end{aligned} \tag{23}$$

These are polystable filtered vector bundles over $\widetilde{S} \setminus \widetilde{Z}$. As for each $k = 0, \cdots, n_L$, $\sigma^* \widetilde{\mathcal{E}}_{*,k} = \widetilde{\mathcal{E}}_{*,k}$, the filtered bundles $\widetilde{\mathcal{E}}_{*,k}$ induce filtered vector bundles $\mathcal{E}_{*,k}$ over $\Sigma \setminus Z$. The flat connections ∇_k^{Lim} will be the unique harmonic unitary connections corresponding to the $\mathcal{E}_{*,k}$. Moreover, for $0 \le k_1 \ne k_2 \le n_L$, based on the definition of D_{k_1} and D_{k_2}, we can always find $\tilde{x} \in \widetilde{Z}_{\mathrm{even}}$, a preimage of an even zero x of q, such that $\widetilde{\mathcal{E}}_{*,k_1}$ and $\widetilde{\mathcal{E}}_{*,k_2}$ have different filtered structures near $\tilde{x}$. Since $\widetilde{S} \to \Sigma$ is not a branched covering over even zeros, $\widetilde{S} \to \Sigma$ we conclude that near x, $\mathcal{E}_{*,k_1}$ and $\mathcal{E}_{*,k_2}$ are different filtered bundles. By [Sim90, Main theorem], the harmonic connections ∇_{k_1} and ∇_{k_2} are not gauge equivalent.

We therefore conclude the following:

Proposition 6.14. *For $0 \le k_1 \ne k_2 \le n_L$, $(\nabla_{k_1}^{\mathrm{Lim}}, \phi_{k_1}^{\mathrm{Lim}})$ and $(\nabla_{k_2}^{\mathrm{Lim}}, \phi_{k_2}^{\mathrm{Lim}})$ are not gauge equivalent in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$.*

Moreover, as with the algebraic compactification of the elements in the $\mathbb{C}^*$ orbit, we would like to compare with the limiting configurations in the space $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$. Over the Dolbeault moduli space $\mathcal{M}_{\mathrm{Dol}}$, there is a natural $\mathbb{Z}_2$ action given by $(\mathcal{E}, \varphi) \to (\mathcal{E}, -\varphi)$, and the fixed point of the $\mathbb{Z}_2$ action is defined to be the real locus of the Dolbeault moduli space, which we denoted by $\mathcal{M}_{\mathrm{Dol}}^{\mathbb{R}}$. It follows from [Hau98, Theorem 6.2] that the source of the orbifold points of the algebraic compactification comes from the quotient of the real locus. Moreover, for $(\mathcal{E}, \varphi) = \chi_{\mathrm{BNR}}(L)$, $(\mathcal{E}, \varphi) \in \mathcal{M}_{\mathrm{Dol}}^{\mathbb{R}}$ if and only if $\sigma^* L = L$.

Given a Higgs bundle $(\mathcal{E}, \varphi)$, under the previous convention we let

$$(\widetilde{\mathcal{E}}_0 = \widetilde{L}_0 \oplus \sigma^* \widetilde{L}_0, \tilde{\varphi}_0 = \mathrm{diag}(\omega, -\omega))$$

be the abelianization of $(\mathcal{E}, \varphi)$. Note that $\sigma^*(\widetilde{\mathcal{E}}_0, \tilde{\varphi}_0) = (\widetilde{\mathcal{E}}_0, \tilde{\varphi}_0)$ and that $\widetilde{L}_0$ is the eigenline bundle for $\tilde{\varphi}_0$ with eigenvalue ω. Therefore, $(\widetilde{\mathcal{E}}_0, \tilde{\varphi}_0)$ is gauge equivalent to $(\widetilde{\mathcal{E}}_0, -\tilde{\varphi}_0)$ if and only if $\widetilde{L}_0 = \sigma^* \widetilde{L}_0$, which is equivalent to saying that $(\mathcal{E}, \varphi)$ lies in the real locus. For the collection of divisors $\mathscr{D} := \{D_1, \cdots, D_{n_L}\}$, for any $D_k \in \mathscr{D}$, we have $\sigma^* D_k \in \mathscr{D}$. Therefore, there exists a permutation of the index $\tau : \{1, \cdots, n_L\} \to \{1, \cdots, n_L\}$ such that $D_{\tau(k)} = \sigma^* D_k$ with $\tau^2 = \mathrm{id}$.

Suppose that for the limiting configurations in (23), $(\widetilde{\mathcal{E}}_{*,k_1}, \varphi_0)$ is gauge equivalent to $(\widetilde{\mathcal{E}}_{*,k_2}, -\varphi_0)$. Then the eigenline bundle for eigenvalue ω will be gauge equivalent, which implies

$$\mathcal{F}_*(\widetilde{L}_0, \chi_{D_{k_1} - \frac{1}{2}\Lambda}) \cong \mathcal{F}_*(\sigma^* \widetilde{L}_0, \chi_{\sigma^* D_{k_2} - \frac{1}{2}\Lambda}).$$

The above equality holds if and only if $\widetilde{L}_0 = \sigma^* \widetilde{L}_0$ and $k_1 = \tau(k_2)$. In summary, we conclude the following:

PROPOSITION 6.15. *Let* $[(\nabla_k^{\mathrm{Lim}}, \phi_k^{\mathrm{Lim}})]$ *be the* $\mathbb{C}^*$ *equivalence class of* $(\nabla_k^{\mathrm{Lim}}, \phi_k^{\mathrm{Lim}})$ *in the space* $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$. *Then the following hold:*

(i) If $(\mathcal{E}, \varphi) \notin \mathcal{M}_{\mathrm{Dol}}^{\mathbb{R}}$, *then for any* $0 \leq k_1 \neq k_2 \leq n_L$, $[(\nabla_{k_1}^{\mathrm{Lim}}, \phi_{k_1}^{\mathrm{Lim}})] \neq [(\nabla_{k_2}^{\mathrm{Lim}}, \phi_{k_2}^{\mathrm{Lim}})]$ *in* $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$.

(ii) If $(\mathcal{E}, \varphi) \in \mathcal{M}_{\mathrm{Dol}}^{\mathbb{R}}$, *then* $[(\nabla_{k_1}^{\mathrm{Lim}}, \phi_{k_1}^{\mathrm{Lim}})] = [(\nabla_{k_2}^{\mathrm{Lim}}, \phi_{k_2}^{\mathrm{Lim}})]$ *in* $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$ *if and only if* $k_1 = k_2$ *or* $k_1 = \tau(k_2)$.

In particular, when $(\mathcal{E}, \varphi) \notin \mathcal{M}_{\mathrm{Dol}}^{\mathbb{R}}$, we obtain $1 + 2n_D$ different $\mathbb{C}^*$ equivalence classes of the limiting configurations in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$, and when $(\mathcal{E}, \varphi) \in \mathcal{M}_{\mathrm{Dol}}^{\mathbb{R}}$, we obtain $1 + n_D$ different $\mathbb{C}^*$ equivalence classes of limiting configurations in $\mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*$.

We define the *analytic Mochizuki map* Υ^{Moc} as

$$\Upsilon^{\mathrm{Moc}} : \mathcal{M}_q \longrightarrow \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}} \ : \ [(\mathcal{E}, \varphi)] \mapsto [(\nabla_0^{\mathrm{Lim}}, \phi_0^{\mathrm{Lim}})], \tag{24}$$

which we recall is the limiting configuration defined by $\Theta^{\mathrm{Moc}}(L)$.

6.4.2 *The continuity of the limiting configurations.* We now introduce the main result of Mochizuki [Moc16]. Fix $(\mathcal{E}, \varphi) = \chi_{\mathrm{BNR}}(L) \in \mathcal{M}_q$. For any real parameter $t > 0$, $(\mathcal{E}, t\varphi)$ is a stable Higgs bundle. By the Kobayashi–Hitchin correspondence, there exists a unique metric H_t that solves the Hitchin equation. Denote by ∇_t the unitary connection defined by H_t, and write $\mathcal{D}_t = \nabla_t + t\phi_t$ for the full $\mathrm{SL}(2, \mathbb{C})$ flat connection. We then have:

THEOREM 6.16. *The family* $(\mathcal{E}, t\varphi)$ *has a unique limiting configuration* $\Upsilon^{\mathrm{Moc}}(\mathcal{E}, \varphi)$ *such that for any compact set* $K \subset \Sigma \setminus Z$,

$$\lim_{t \to \infty} |(\nabla_t, \phi_t) - \Upsilon^{\mathrm{Moc}}(\mathcal{E}, \varphi)|_{\mathcal{C}^l(K)} = 0.$$

Moreover, if write $(\mathcal{E}, \varphi) = \chi_{\mathrm{BNR}}(L)$ *and suppose that* $L = p_* \widetilde{L}$, *then there exist t-independent positive constants* $C_{l,K}$ *and* $C'_{l,K}$ *such that*

$$|(\nabla_t, \phi_t) - \Upsilon^{\mathrm{Moc}}(\mathcal{E}, \varphi)|_{\mathcal{C}^l(K)} \leq C_{l,K} e^{-C'_{l,K} t}.$$

Because the map Υ^{Moc} is the composition of $\Theta^{\mathrm{Moc}} \circ \chi_{\mathrm{BNR}}^{-1}$ with the nonabelian Hodge correspondence, the behavior of Υ^{Moc} is the same as Θ^{Moc}. Recall the decomposition $\mathcal{M}_q = \bigcup \mathcal{M}_{q,D}$ from the end of Section 5. By Theorem 6.11, Proposition 6.12, Proposition 6.14 and Proposition 6.15, we obtain:

THEOREM 6.17. *Let* q *be an irreducible quadratic differential. The map* $\Upsilon^{\mathrm{Moc}} : \mathcal{M}_q \to \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$ *satisfies the following properties:*

(i) if all the zeros of q *are odd, then* Υ^{Moc} *is continuous;*

(ii) if at least one zero of q *is even, then for each* $(\mathcal{F}, \psi) \in \mathcal{M}_{q,D}$, *there exists an integer* $2n_D$ *that depends only on* D *and* $2n_D$ *sequences* $\{(\mathcal{E}_i^k, \varphi_i^k)\}$ *for* $k = 1, \ldots, 2n_D$, *such that*

- $\lim_{i \to \infty} (\mathcal{E}_i^k, \varphi_i^k) = (\mathcal{F}, \psi)$ *for* $k = 1, \ldots, 2n_D$;
- *and if we write*

$$\eta^k := \lim_{i \to \infty} \Upsilon^{\mathrm{Moc}}(\mathcal{E}_i^k, t_i \varphi_i^k) \quad , \quad \xi := \lim_{i \to \infty} \Upsilon^{\mathrm{Moc}}(\mathcal{F}, t_i \psi)$$

  - *if* $(\mathcal{F}, \psi)$ *doesn't lie in the real locus, then* $\xi, \eta^1, \ldots, \eta^{2n_D}$ *are* $2n_D + 1$ *different limiting configurations;*
  - *if* $(\mathcal{F}, \psi)$ *lies in the real locus, then* $\eta^i \cong \eta^{n_D + i}$ *for* $i = 1, \cdots, n$, *and we obtain* $n_D + 1$ *different limiting configurations.*

7. Reducible singular fibre and the Mochizuki map

We now investigate properties of the Hitchin fibre associated with a reducible quadratic differential, as discussed in [GO13]. Additionally, we provide an overview of Mochizuki's technique for constructing limiting configurations of Hitchin fibres for reducible quadratic differentials, as detailed in [Moc16]. We also analyse the continuity of the Mochizuki map.

7.1 Local description of a Higgs bundle

Write $q = -\omega \otimes \omega$ with $\omega \in H^0(K)$, $\Lambda = \mathrm{Div}(\omega)$, $Z = \mathrm{supp}\,(\Lambda)$ and $\mathcal{M}_q = \mathcal{H}^{-1}(q)$. Compared to the irreducible case, $\mathcal{M}_q$ contains strictly semistable Higgs bundles, so we let $\mathcal{M}_q^{\mathrm{st}}$ denote the stable locus. We point out that there is a sign ambiguity in the choice of ω, which actually plays an important role in the following:

7.1.1 *Local description.* Given a Higgs bundle $(\mathcal{E}, \varphi)$ with $\det(\varphi) = q$, define the line bundles

$$L_\pm := \ker(\varphi \pm \omega). \tag{25}$$

Then the inclusion maps $L_\pm \to \mathcal{E}$ are injective. Similarly, we may define an abelianisation of $(\mathcal{E}, \varphi)$ by $(\mathcal{E}_0 = L_+ \oplus L_-, \varphi_0 = \mathrm{diag}(\omega, -\omega))$. We then have the natural inclusion $\iota : \mathcal{E}_0 \to \mathcal{E}$, which is an isomorphism on $\Sigma \setminus Z$, and $\varphi \circ \iota = \iota \circ \varphi_0$.

It follows from [GO13, Prop. 7.10] that $L_\pm$ are the only φ-invariant subbundles of $\mathcal{E}$. If we write $d_\pm := \deg(L_\pm)$, then $(\mathcal{E}, \varphi)$ is stable (resp. semistable) if and only if $d_\pm < 0$ (resp. ≤ 0). As $\det(\mathcal{E}) = \mathcal{O}$, the map $\det(\iota) : L_+ \otimes L_- \to \mathcal{O}$ defines a divisor $D = \mathrm{Div}(\det(\iota))$ such that $L_+ \otimes L_- = \mathcal{O}(-D)$. Therefore, we obtain

$$d_+ + d_- + \deg D = 0$$

and $0 \leq D \leq \Lambda$. The Higgs bundle $(\mathcal{E}, \varphi)$ is semistable if and only if $-\deg D \leq d_+ \leq 0$ and is stable if the equalities are strict. For the rest of this section, we will always write $D = \sum_{p \in Z} \ell_p p$.

By $\mathcal{M}_{q,D}$ we mean the set of Higgs bundles $(\mathcal{E}, \varphi) \in \mathcal{M}_q$ for which the relation $L_+ \otimes L_- = \mathcal{O}(-D)$ holds. Consequently, we have $\mathcal{M}_q = \bigcup_{0 \leq D \leq \Lambda} \mathcal{M}_{q,D}$.

7.1.2 *Semistable settings.* As the fibre $\mathcal{M}_q$ might contain strictly semistable Higgs bundles, we now explicitly enumerate all of the possible S-equivalence classes. When $D = 0$, then $L_- = L_+^{-1}$ and $\deg(L_+) = 0$. The corresponding Higgs bundle is polystable and can be explicitly written as

$$\left(L \oplus L^{-1}, \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \right)$$

where $L \in \mathrm{Jac}(\Sigma)$. When $D \neq 0$, suppose $\deg(L_+) = -\deg(D)$. Then $L_- = L_+^{-1}(-D)$ and $\deg(L_-) = 0$. Under S-equivalence, the polystable Higgs bundle is

$$\left(L_+(D) \oplus L_+^{-1}(-D), \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \right)$$

where $L_+ \in \mathrm{Pic}^{-\deg(D)}(\Sigma)$.

7.2 Reducible spectral curves

In this subsection, we introduce the algebraic data in [GO13], which describes the singular fibre with a reducible spectral curve. This plays a similar role to the parabolic modules. See [GO13, Sec. 7.1] for more details.

For any effective divisor D and line bundle L, define the space

$$H^0(D,L) = \bigoplus_{p\in \operatorname{supp} D} \mathcal{O}(L)_p/\sim$$

where $s_1 \sim s_2$ if and only if $\operatorname{ord}_p([s_1]-[s_2]) \geq D_p$. Let $L \in \operatorname{Pic}(\Sigma)$, and define the following subspaces of $H^0(\Lambda, L^2K)$:

$$\mathcal{V}(D,L) := \{s \in H^0(\Lambda, L^2K) \mid \operatorname{ord}_p(s) = \Lambda_p - D_p, \text{ if } D_p > 0;\ s|_p = 0, \text{ if } D_p = 0\};$$
$$\mathcal{W}(D,L) = \{s \in H^0(\Lambda, L^2K) \mid s|_{\operatorname{supp}(\Lambda - D)} = 0\}.$$

One checks that $\mathcal{W}(D,L) = \cup_{D' \leq D} \mathcal{V}(D',L)$. Moreover, the space $\mathcal{V}(D,L)$ is a linear subspace of $H^0(\Lambda, L^2K)$ with a hyperplane removed. In addition, $\mathbb{C}^*$ acts on $\mathcal{V}(D,L)$ by multiplication, and $\dim(\mathcal{V}(D,L)/\mathbb{C}^*) = \deg(D) - 1$.

We define the fibrations

$$p_m : \mathscr{V}(D,m) \longrightarrow \operatorname{Pic}^m(\Sigma),\ p_m : \mathscr{W}(D,m) \longrightarrow \operatorname{Pic}^m(\Sigma)$$

such that for $L \in \operatorname{Pic}^m(\Sigma)$, the fibres are $\mathcal{V}(D,L)$ and $\mathcal{W}(D,L)$.

7.2.1 *Algebraic data from the extension.* The Higgs bundle $(\mathcal{E}, \varphi)$ can be understood in terms of an extension. Since $\det(\mathcal{E}) = \mathcal{O}$, we have the exact sequence

$$0 \longrightarrow L_+ \longrightarrow \mathcal{E} \longrightarrow L_+^{-1} \longrightarrow 0.$$

For each $p \in Z$, with $U \subset \Sigma$ a neighbourhood of p, $(\mathcal{E}, \varphi)$ can be written in terms of a splitting of $\mathcal{C}^\infty$ bundles

$$\mathcal{E} = L_+ \oplus_{\mathcal{C}^\infty} L_+^{-1}\ \ \bar{\partial}_{\mathcal{E}} = \begin{pmatrix} \bar{\partial}_{L_+} & b \\ 0 & \bar{\partial}_{L_+^{-1}} \end{pmatrix},\ \ \varphi = \begin{pmatrix} \omega & c \\ 0 & -\omega \end{pmatrix}.$$

Now consider $\varphi|_\Lambda$. Because the induced morphisms

$$(L+)|_\Lambda \longrightarrow (L_+ \otimes K)|_\Lambda\ \ (\mathcal{E}/L_+)|_\Lambda \longrightarrow (\mathcal{E}/L_+ \otimes K)|_\Lambda$$

both vanish, we obtain the map $s : L_+^{-1}|_\Lambda \simeq (\mathcal{E}/L_+)|_\Lambda \to L_+K|_\Lambda$ or, equivalently, a section $s \in H^0(\Lambda, L_+^2K)$. Moreover, by [GO13, Lemma 7.12], $\operatorname{Div}(s) = \Lambda - D$, where $\operatorname{Div}(s)$ is the divisor defined by zeros of s. Therefore, given any $(\mathcal{E}, \varphi) \in \mathcal{M}_q$, we obtain an $L \in \operatorname{Pic}^m(\Sigma)$ and an element in $\mathcal{V}(D,L)$. The stability condition implies that if $0 \leq D \leq \Lambda$, we have $-\deg D \leq \deg L \leq 0$.

7.2.2 *Inverse construction.* The inverse of the construction above also holds; for further details, see [GO13, Sec. 7] and [Hor22a, Sec. 5]. Given $L \in \operatorname{Pic}^m(\Sigma)$ and $s \in \mathcal{V}(D,L)$, we define a Higgs bundle via extensions as follows: From $q = -\omega \otimes \omega, L$, we have a short exact sequence of complexes of sheaves:

$$\begin{array}{ccccccccc} & & C_1^* & & C_2^* & & C_3^* & & \\ 0 & \longrightarrow & L^2 & \xrightarrow{=} & L^2 & \xrightarrow{\mathrm{pr}} & 0 & \longrightarrow & 0\ , \\ & & \downarrow \mathrm{id} & & \downarrow c & & \downarrow 0 & & \\ 0 & \longrightarrow & L^2 & \xrightarrow{c} & L^2K & \xrightarrow{\mathrm{res}(\Lambda)} & L^2K|_\Lambda & \longrightarrow & 0 \end{array}$$

where, for a section $s' \in \Gamma(L^2)$, $c(s') := \sqrt{-1}\omega s'$, and $\operatorname{res}(\Lambda)$ is the restriction map to the divisor Λ. The long exact sequence in hypercohomology implies that $\operatorname{res}(\Lambda)$ induces an isomorphism

$$\operatorname{res}(\Lambda) : \mathbf{H}^1(C_2^*) \cong \mathbf{H}^1(C_3^*) = H^0(\Lambda, L^2K).$$

Moreover, $\mathbf{H}^1(C_2^*)$ fits into an exact sequence

$$0 \longrightarrow W_1 \longrightarrow \mathbf{H}^1(C_2^*) \longrightarrow W_2 \longrightarrow 0$$

where

$$W_1 = \operatorname{coker}\left(c : H^0(L^2) \longrightarrow H^0(L^2K)\right)$$
$$W_2 = \ker\left(c : H^1(L^2) \longrightarrow H^1(L^2K)\right).$$

Now $H^1(\Sigma, L^2)$ parameterises extensions

$$0 \longrightarrow L \longrightarrow \mathcal{E} \longrightarrow L^{-1} \longrightarrow 0.$$

Given $b \in W_2$, we can find $c' \in \Gamma(L^2K)$, $\bar\partial c' = 2b\omega$ and construct the Higgs bundle

$$E = L \oplus_{\mathcal{C}^\infty} L^{-1},\ \bar\partial_E = \begin{pmatrix} \bar\partial_L & b \\ 0 & \bar\partial_{L^{-1}} \end{pmatrix},\ \varphi = \begin{pmatrix} \omega & c' \\ 0 & -\omega \end{pmatrix}. \tag{26}$$

For $0 \le D \le \Lambda$ and $-\deg D \le m \le 0$, the construction above defines the map

$$\wp : \mathscr{V}(D, m) \longrightarrow \mathcal{M}_q \ \ s \in \mathcal{V}(D, L) \mapsto [(\mathcal{E}, \varphi)]$$

where $[(\mathcal{E}, \varphi)]$ is the S-equivalence class of the Higgs bundle constructed in (26) (note that for $(b, c') \neq (0, 0)$, the orbit of $(\mathcal{E}, \varphi)$ is closed in the semistable locus if and only if $\deg(L) \neq 0$). When $D = 0$, $\mathcal{V}(\Lambda, L) = \{0\}$ and the image of $\wp : \mathscr{V}(\Lambda, 0) \to \mathcal{M}_q$ consists of the polystable Higgs bundles $\mathcal{E} = L \oplus L^{-1}$, $\varphi = \operatorname{diag}(\omega, -\omega)$ such that $L^2 \cong \mathcal{O}_\Sigma$.

Theorem 7.1. *For $0 \le D \le \Lambda$ and $-\deg(D) \le m_1 \le 0$ and the map $\wp : \mathscr{V}(D, m_1) \to \mathcal{M}_q$, we have*

- *(i) for $m_2 = -\deg(D) - m_1$, we have $\wp(\mathscr{V}(D, m_1)) = \wp(\mathscr{V}(D, m_2))$;*
- *(ii) for the $\mathbb{C}^*$ action on $\mathscr{V}(D, m_1)$ by multiplication, for $\xi \in \mathscr{V}(D, m_1)$, $\wp(\mathbb{C}^*\xi) = \wp(\xi)$;*
- *(iii) when $m_1 \neq -\frac{1}{2}\deg(D), 0, -\deg(D)$, $\wp : \mathscr{V}(D, m_1)/\mathbb{C}^* \to \mathcal{M}_q$ is an isomorphism onto its image;*
- *(iv) when $m_1 = -\frac{1}{2}\deg(D)$, $\wp : \mathscr{V}(D, m_1)/\mathbb{C}^* \to \mathcal{M}_q$ is a double-branched covering, which is branched along line bundles $L \in \operatorname{Pic}^{m_1}(\Sigma)$ such that $L^2 \cong \mathcal{O}(-D)$;*
- *(v) when $D = 0$, then $\wp : \mathscr{V}(\Lambda, 0) \to \mathcal{M}_q$ is a double-branched covering, which is branched along $L \in \operatorname{Pic}^0(\Sigma)$ such that $L^2 \cong \mathcal{O}$;*
- *(vi) when $m_1 = 0, -\deg(D)$, $\wp : \mathscr{V}(\Lambda, 0) \to \mathcal{M}_q$ is surjective but not injective. The image of $\wp$ are all polystable Higgs bundles.*

Remark 7.2. Parts (*iii*) and (*vi*) of Theorem 7.1 are different from the statements in [GO13, Theorem 7.7]. Because of the S-equivalence, when $m_1 = 0, -\deg(D)$, the map $\wp$ will not be injective. We thank the authors of [GO13] for clarification of this point.

Example 7.3. When $g = 2$ for $q = -\omega \otimes \omega$, we can write $\Lambda = p_1 + p_2$ or $\Lambda = 2p$. In either case, the $\mathcal{M}_q^{\mathrm{st}} = \wp(\mathscr{V}(D, m))$ for $-\deg(D) < m < 0$ and $0 \le D \le \Lambda$. Therefore, $m = -1$, $D = \Lambda$ and $\wp(\mathscr{V}(\Lambda, -1)) = \mathcal{M}_q^{\mathrm{st}}$. Moreover, generically, the map $\wp : (\mathscr{V}(\Lambda, -1))/\mathbb{C}^* \to \mathcal{M}_q^{\mathrm{st}}$ is 2-to-1.

7.3 The stratification of the singular fibre

We now present two stratifications of $\mathcal{M}_q$. Recall that from any Higgs bundle $(\mathcal{E}, \varphi)$, we obtain two line bundles, $L_\pm$, and a divisor, D. There are two different stratifications: one given by the divisor, D, and the other given by the degree of L_+.

7.3.1 *Divisor stratification.* We first discuss the stratification defined by the divisor. Indeed, using D, decompose into strata: $\mathcal{M}_q = \bigcup_{0 \le D \le \Lambda} \mathcal{M}_{q,D}$. Because the definition of $L_\pm$ depends on the choice of the square root, there is no natural map from $\mathcal{M}_D$ to $\mathrm{Pic}(\Sigma)$. Consider the following space: $\mathbb{V}_D = \bigcup_{-\deg(D) \le m \le 0} \mathscr{V}(D, m)$. This forms a fibration:

$$\tau : \mathbb{V}_D \longrightarrow \bigcup_{-\deg(D) \le m \le 0} \mathrm{Pic}^m(\Sigma).$$

Moreover, for $L \in \mathrm{Pic}^m(\Sigma)$, we have $\tau^{-1}(L) = \mathscr{V}(D, L)$ and $\dim(\tau^{-1}(L)/\mathbb{C}^*) = \deg(D) - 1$. By Theorem 7.1, $\wp|_{\mathbb{V}_D} : \mathbb{V}_D \to \mathcal{M}_D$ is surjective. Since

$$\wp|_{\mathscr{V}(D,m)} = \wp|_{\mathscr{V}(D, -\deg(D) - m)}$$

generically, $\wp|_{\mathbb{V}_D}$ is a 2-to-1 map.

In summary, we obtain the following map, which characterises the singular fibre.

$$\wp : \mathbb{V} = \bigcup_{0 \le D \le \Lambda} \mathbb{V}_D \to \mathcal{M}_q = \bigcup_{0 \le D \le \Lambda} \mathcal{M}_{q,D}.$$

The top stratum is given by $D = \Lambda$.

7.3.2 *Degree stratification.* We next introduce the stratification defined by degrees; this encodes how different divisor stratifications are glued together. For $-(2g-2) \le m \le 0$ and $L \in \mathrm{Pic}^m(\Sigma)$, define $\mathbb{W}(L) := \bigcup_{\deg D \ge -m} \mathcal{V}(D, L)$. This set is connected, based on the definition and on [GO13, Lemma 7.14]. Moreover, if we define

$$\mathbb{W}_m := \bigcup_{-m \le \deg D,\, 0 \le D \le \Lambda} \mathscr{V}(D, m) \quad \mathbb{W} := \bigcup_{-(2g-2) \le m \le 0} \mathbb{W}_m$$

then we have $\wp(\mathbb{W}) = \wp(\mathbb{V})$. We should also note that although $\mathbb{W}_m \cap \mathbb{W}_n = \emptyset$ for any $m \ne n$, $\mathbb{W}$ is connected. As L_+, L_- are symmetric, by Theorem 7.1 we have

$$\wp(\mathscr{V}(D, m)) = \wp(\mathscr{V}(D, -\deg(D) - m))$$

which implies that for any integer $-(2g - 2 + m) \le n \le 0$, $\wp\mathbb{W}_m \cap \wp\mathbb{W}_n \ne \emptyset$.

We now give an example of the degree stratification when $g = 2$.

Example 7.4. Suppose ω has only one zero with order 2. Then $\Lambda = 2p$, and all possible divisors are $D_2 = 2p, D_1 = p, D_0 = 0$. The degree stratification is

$$\mathbb{W}_{-2} = \mathscr{V}(D_2, -2) \quad \mathbb{W}_{-1} = \mathscr{V}(D_2, -1) \cup \mathscr{V}(D_1, -1);$$
$$\mathbb{W}_0 = \mathscr{V}(D_0, 0) \cup \mathscr{V}(D_1, 0) \cap \mathscr{V}(D_2, 0).$$

The image of $\wp(\mathscr{V}(D_2, -1))$ is stable; $\wp(\mathscr{V}(D_0, 0))$ is polystable; and $\wp(\mathscr{W} \setminus (\mathscr{V}(D_2, -1) \cup \mathscr{V}(D_0, 0)))$ is semistable.

Moreover, we have $\wp(\mathscr{V}(D_2, -2)) = \wp(\mathscr{V}(D_2, 0))$ and $\wp(\mathscr{V}(D_1, -1)) = \wp(\mathscr{V}(D_1, 0))$, and $\wp|_{\mathscr{V}(D_2,-1)}$ is a branched covering. Furthermore, we have $\wp(\mathscr{V}(D_2, -1)) \cap \wp(\mathscr{V}(D_1, 0)) \ne 0$ and $\wp(\mathscr{V}(D_2, -1)) \cap \wp(\mathscr{V}(D_0, 0)) = 0$.

7.4 Algebraic Mochizuki map

Based on the study of the local rescaling properties of Higgs bundles, Mochizuki introduced a weight for each $p \in Z$ in [Moc16, Sec. 3]. To be more specific, let c be a real number. For each $p \in Z$, the weight we consider is given by

$$\chi_p(c) = \min\{\ell_p, (m_p+1)c + \ell_p/2\}$$

where $\mathrm{Div}(\omega) = \sum_p m_p p$ and ℓ_p is defined as in Section 7.1.1.

By utilizing the global geometry of a Higgs bundle, we can uniquely determine the constant c. We aim to choose the sign of ω such that $d_+ \leq d_-$.

Lemma 7.4 [Moc16, Lemma 4.3].
If $(\mathcal{E}, \varphi)$ is stable, then there exists a unique constant $c \geq 0$ such that

$$d_+ + \sum_{p\in Z} \chi_p(c) = 0 \ \ d_- + \sum_{p\in Z} (\ell_p - \chi_p(c)) = 0.$$

Proof. Since $(\mathcal{E}, \varphi)$ is stable, we have $-\sum \ell_p < d_\pm < 0$. We define the function

$$f(c) = d_+ + \sum_p \chi_p(c) \tag{27}$$

which is strictly increasing. Moreover, for c sufficiently large, $\chi_p(c) = \ell_p$, and therefore $f(c) = d_+ + \sum_p \ell_p = -d_- > 0$. Additionally, $f(0) = d_+ + \sum_p(\ell_p/2)$. Since $d_+ \leq d_-$ and $d_+ + d_- + \sum_p \ell_p = 0$, we obtain $f(0) \leq 0$. The monotonicity of f implies the existence of c_0 such that $f(c_0) = 0$. □

From this construction, if $d_+ \leq d_-$, two weighted bundles $(L_+, \chi_p(c_0))$ and $(L_-, \ell_p - \chi_p(c_0))$ are obtained with weights $\chi_p(c_0)$ and $\ell_p - \chi_p(c_0)$ at each $p \in Z$, respectively. On the other hand, if $d_+ \geq d_-$, by symmetry, weighted bundles $(L_+, \ell_p - \chi_p(c_0))$ and $(L_-, \chi_p(c_0))$ are obtained. When $(\mathcal{E}, \varphi)$ is strictly semistable, S-equivalent to $(L, \omega) \oplus (L^{-1}, -\omega)$, then we would like to consider the weighted bundles $(L, 0) \oplus (L^{-1}, 0)$ with weight zero.

Next, we define the algebraic Mochizuki map. Let $\mathscr{F}_\pm(\Sigma)$ be the space of rank 1 degree zero-filtered bundles on Σ, and let $\mathscr{F}_2(\Sigma) := \mathscr{F}_+(\Sigma) \times \mathscr{F}_-(\Sigma)$ be the direct product. Fix a choice of ω. Then from any Higgs bundle $(\mathcal{E}, \varphi)$, we obtain the subbundles $L_\pm$ with degree $d_\pm$ and define the algebraic Mochizuki map:

$$\Theta^{\mathrm{Moc}} : \mathcal{M}_q \longrightarrow \mathscr{F}_2(\Sigma),$$
$$\Theta^{\mathrm{Moc}}(\mathcal{E}, \varphi) := \begin{cases} \mathcal{F}_*(L_+, \chi_p(c_0)) \oplus \mathcal{F}_*(L_-, \ell_p - \chi_p(c_0)), \text{ if } d_+ \leq d_- \\ \mathcal{F}_*(L_+, \ell_p - \chi_p(c_0)) \oplus \mathcal{F}_*(L_-, \chi_p(c_0)), \text{ if } d_- \leq d_+ \end{cases}, \ (\mathcal{E}, \varphi) \text{ stable},$$
$$\Theta^{\mathrm{Moc}}(\mathcal{E}, \varphi) := \mathcal{F}_*(L, 0) \oplus \mathcal{F}_*(L^{-1}, 0), \ (\mathcal{E}, \varphi) \text{ semistable}.$$

We list some properties of this map:

Proposition 7.5. *For Θ^{Moc}, we have:*

(i) for each $\mathscr{V}(D, m)$ with $0 \leq D \leq \Lambda$, $-\deg(D) \leq m \leq 0$, $\Theta^{\mathrm{Moc}}|_{\wp(\mathscr{V}(D,m))}$ is continuous;

(ii) for $i = 1, 2$ and $s_i \in \mathbb{V}_D$ with $(\mathcal{E}_i, \varphi_i) := \wp(s_i)$, suppose $\tau(s_1) = \tau(s_2)$; then $\Theta^{\mathrm{Moc}}(\mathcal{E}_1, \varphi_1) = \Theta^{\mathrm{Moc}}(\mathcal{E}_2, \varphi_2)$. In particular, Θ^{Moc} is not injective.

Proof. The proof follows directly from the definition. □

A Higgs bundle $(\mathcal{E},\varphi)\in\mathcal{M}_q$ is called 'exotic' if the constant c in Lemma 7.5 satisfies $c\neq 0$. This new behavior appears only in the Hitchin fibre with a reducible spectral curve.

Proposition 7.7. *A Higgs bundle $(\mathcal{E},\varphi)$ is not exotic if and only if its corresponding degrees satisfy $d_+=d_-$.*

Proof. This is straightforward from the definition and from Lemma 7.4. □

7.5 Discontinuous behavior

In this subsection, we study the discontinuous behavior of Θ^{Moc}. Consider a sequence of algebraic data $(L_i,q_i)\in\mathbb{W}_m$, where $L_i\in\mathrm{Pic}^m$ and $q_i\in\mathscr{V}(D,L_i)$. We assume that $\lim_{i\to\infty}L_i=L_\infty$ in Pic^m and $\lim_{i\to\infty}q_i=q_\infty\in\mathscr{V}(D_\infty,L)$, for $D_\infty\neq D$. As the space $\bigcup_{\deg D'\geq -m}\mathscr{V}(D',m)$ is connected, we can always find such a sequence.

Let $L_+^i:=L_i$ and $L_-^i:=L_i^{-1}\otimes\mathcal{O}(-D)$. By Lemma 7.5 the weight function, which we denote by $\chi_\pm$, is independent of i. In addition, we have

$$\lim_{i\to\infty}\Theta^{\mathrm{Moc}}\circ\wp(L_i,q_i)=\mathcal{F}_*(L_\infty,\chi_+)\oplus\mathcal{F}_*(L_\infty^{-1}(-D),\chi_-).$$

For $(L_\infty,q_\infty\in\mathscr{V}(D_\infty,L))$, let $\chi_\pm^\infty$ be the corresponding weights. These depend on D_∞ and m. Then

$$\Theta^{\mathrm{Moc}}\circ\wp(L_\infty,q_\infty)=\mathcal{F}_*(L_\infty,\chi_+^\infty)\oplus\mathcal{F}_*(L_\infty^{-1}\otimes\mathcal{O}(-D_\infty),\chi_-^\infty).$$

Therefore, we obtain

$$\begin{aligned}&\lim_{i\to\infty}\Theta^{\mathrm{Moc}}\circ\wp(L_i,q_i)\\&=\Theta^{\mathrm{Moc}}\circ\wp(L_\infty,q_\infty)\otimes(\mathcal{F}_*(\mathcal{O},\chi_+-\chi_+^\infty)\oplus\mathcal{F}_*(\mathcal{O}(D_\infty-D),\chi_-\chi_-^\infty)).\end{aligned}\tag{28}$$

Proposition 7.7. *When $g\geq 3$, there exists a sequence $(\mathcal{E}_i,\varphi_i)\in\mathcal{M}_q$ of stable Higgs bundles with stable limit $(\mathcal{E}_\infty,\varphi_\infty)=\lim_{i\to\infty}(\mathcal{E}_i,\varphi_i)$ such that*

$$\lim_{i\to\infty}\Theta^{\mathrm{Moc}}(\mathcal{E}_i,\varphi_i)\neq\Theta^{\mathrm{Moc}}(\mathcal{E}_\infty,\varphi_\infty).$$

Proof. Choose $D=\Lambda$ and $d_+=-(g-1)$ with $L_i=L\in\mathrm{Pic}^{d_+}(\Sigma)$, and study the degenerate behavior for a family $q_i\in\mathcal{V}(\Lambda,L)$ that converges to $q_\infty\in\mathscr{V}(D_\infty,L)$. Here, D_∞ satisfies $D_\infty\leq D$ and $\deg(D_\infty)=\deg(D)-1$. As q_i lies in the top stratum, we can always find such a family. Take $(\mathcal{E}_i,\varphi_i)=\wp(L_i,q_i)$ and $(\mathcal{E}_\infty,\varphi_\infty)=\wp(L,q_\infty)$. When $g\geq 3$, we have $-\deg(D_\infty)<d_+\leq -\frac{1}{2}\deg(D_\infty)$, which implies that $(\mathcal{E}_\infty,\varphi_\infty)$ is a stable Higgs bundle.

Write $D=\sum_p\ell_p$. As $(\mathcal{E}_i,\varphi_i)$ is nonexotic, the weights will be $\chi_+(p)=\chi_-(p)=\ell_p/2$. However, as $\deg(D_\infty)\neq 2d_+$, $(\mathcal{E}_\infty,\varphi_\infty)$ is exotic. By Proposition 7.6, if we write $\chi_\pm^\infty(p)$ for the weight functions with corresponding constant c, then $c>0$. Therefore, for $p\neq p_0$, we have $\chi_+^\infty(p)=(m_p+1)c+m_p/2>m_p/2=\chi_+(p)$. By (28), $\lim_{i\to\infty}\Theta^{\mathrm{Moc}}(\mathcal{E}_i,\varphi_i)\neq\Theta^{\mathrm{Moc}}(\mathcal{E}_\infty,\varphi_\infty)$. □

When $g=2$, the stratification is simpler, and we have the following:

Proposition 7.8. *When $g=2$, the following holds:*

(i) Suppose $\Lambda=p_1+p_2$ for $p_1\neq p_2$; then $\Theta^{\mathrm{Moc}}|_{\mathcal{M}_q^{\mathrm{st}}}$ is continuous. Moreover, there exists a sequence of stable Higgs bundles $(\mathcal{E}_i,\varphi_i)\in\mathcal{M}_q$ where the limit $(\mathcal{E}_\infty,\varphi_\infty)=\lim_{i\to\infty}(\mathcal{E}_i,\varphi_i)$ is semistable and where $\gamma(0)$ is also semistable; furthermore

$$\lim_{i\to\infty}\Theta^{\mathrm{Moc}}(\mathcal{E}_i,\varphi_i)\neq\Theta^{\mathrm{Moc}}(\mathcal{E}_\infty,\varphi_\infty).$$

(ii) Suppose $\Lambda=2p$. Then $\Theta^{\mathrm{Moc}}|_{\mathcal{M}_q^{\mathrm{st}}}$ is continuous.

Proof. For (i), suppose $\Lambda = p_1 + p_2$; then by Example 7.2, we have $\mathcal{M}_q^{\mathrm{st}} = \wp(\mathscr{V}(\Lambda, -1))$. By Proposition 7.6, $\Theta_q^{\mathrm{Moc}}|_{\mathcal{M}_q^{\mathrm{st}}}$ is continuous. However, for semistable elements, other strata must be taken into consideration. Take $L \in \mathrm{Pic}^{-1}(\Sigma)$ and $q_i \in \mathcal{V}(\Lambda, L)$ such that q_i converges to $q_\infty \in \mathcal{V}(p_1, L)$. We define $(\mathcal{E}_i, \varphi_i) = \wp(L, q_i)$ and $(\mathcal{E}_\infty, \varphi_\infty) = \wp(L, q_\infty)$. For each i,

$$\Theta^{\mathrm{Moc}}(\mathcal{E}_i, \varphi_i) = \mathcal{F}_*(L, (\tfrac{1}{2}, \tfrac{1}{2})) \oplus \mathcal{F}_*(L^{-1}(-\Lambda), (\tfrac{1}{2}, \tfrac{1}{2})).$$

Moreover, we have

$$\Theta^{\mathrm{Moc}}(\mathcal{E}_\infty, \varphi_\infty) = \mathcal{F}_*(L(D), (0,0)) \oplus \mathcal{F}_*(L^{-1}(-D), (0,0)) \neq \lim_{i\to\infty} \Theta^{\mathrm{Moc}}(\mathcal{E}_i, \varphi_i).$$

For (ii), by Example 7.3, $\wp(\mathscr{V}(D_2, -1)) = \mathcal{M}_q^{\mathrm{st}}$, and by Proposition 7.5, $\Theta_q^{\mathrm{Moc}}|_{\mathcal{M}_q^{\mathrm{st}}}$ is continuous. We now consider the behavior of the filtered bundle when crossing the divisors. □

7.6 The analytic Mochizuki map and limiting configurations

In this subsection, we construct the analytic Mochizuki map for the Hitchin fibre with a reducible spectral curve. We also introduce the convergence theorem of Mochizuki as stated in [Moc16] and examine the discontinuous behavior of the analytic Mochizuki map.

For $(\mathcal{E}, \varphi) \in \mathcal{M}_q$, we can express the abelianization as $(\mathcal{E}_0, \varphi_0) = \left(L_+ \oplus L_-, \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}\right)$;

thus $\Theta^{\mathrm{Moc}}(\mathcal{E}, \varphi) = \mathcal{F}_*(L_+, \chi_+) \oplus \mathcal{L}_-(L_-, \chi_-) \in \mathcal{F}_2(\Sigma)$. Via the nonabelian Hodge correspondence for filtered bundles, we obtain two Hermitian metrics $h_\pm^{\mathrm{Lim}}$ with corresponding Chern connections $A_{h_\pm^{\mathrm{Lim}}}$. These metrics satisfy the following proposition:

Proposition 7.9. *The metrics $h_\pm^{\mathrm{Lim}}$ over $L_\pm$ satisfy*

- *i) $F_{A_{h_\pm^{\mathrm{Lim}}}} = 0$ and $h_+^{\mathrm{Lim}} h_-^{\mathrm{Lim}} = 1$;*
- *ii) for every $p \in \Sigma$, there exists an open neighbourhood (U, z) with $P = \{z = 0\}$ such that $|z|^{-2\chi_p(c_0)} h_+^{\mathrm{Lim}}$ and $|z|^{2\chi_p(c_0) + 2l_P} h_-^{\mathrm{Lim}}$ extend smoothly to $L_\pm|_U$.*

Now, $H^{\mathrm{Lim}} := h_+^{\mathrm{Lim}} \oplus h_-^{\mathrm{Lim}}$ is a metric on $\mathcal{E}_0$ that induces a metric on $(\mathcal{E}, \varphi)|_{\Sigma\setminus Z}$ because $(\mathcal{E}, \varphi)|_{\Sigma\setminus Z} \cong (\mathcal{E}_0, \varphi_0)|_{\Sigma\setminus Z}$. Let $(A^{\mathrm{Lim}}, \phi^{\mathrm{Lim}})$ be the Chern connection defined by $(\mathcal{E}, \varphi, H^{\mathrm{Lim}})$ over $\Sigma \setminus Z$. Then $(A^{\mathrm{Lim}}, \phi^{\mathrm{Lim}})$ is a limiting configuration that satisfies the decoupled Hitchin equations (9). The analytic Mochizuki map Υ^{Moc} is defined as

$$\Upsilon^{\mathrm{Moc}} : \mathcal{M}_q \longrightarrow \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}, \quad \Upsilon^{\mathrm{Moc}}(\mathcal{E}, \varphi) = (A^{\mathrm{Lim}}, \phi^{\mathrm{Lim}}). \tag{29}$$

Note that H^{Lim} is not unique: For any constant c, the metric $ch_+^{\mathrm{Lim}} \oplus c^{-1} h_-^{\mathrm{Lim}}$ defines the same Chern connection as H^{Lim}. In any case, the map Υ^{Moc} is well-defined.

Suppose $(\mathcal{E}, \varphi)$ is an S-equivalence class of a semistable Higgs bundle. Let H_t be the harmonic metric for $(\mathcal{E}, t\varphi)$. For each constant $C > 0$, define μ_C to be the automorphism of $L_+ \oplus L_-$ given by $\mu_C = C\mathrm{id}_{L_+} \oplus C^{-1}\mathrm{id}_{L_-}$. As $\mathcal{E} \cong L_+ \oplus L_-$ on $\Sigma \setminus Z$, $\mu_C^* H_t$ can be regarded as a metric on $\mathcal{E}|_{\Sigma\setminus Z}$. Take any point $x \in \Sigma \setminus Z$ and a frame e_x of $L_+|x$, and define

$$C(x,t) := \left(\frac{h_{L_+}^{\mathrm{Lim}}(e_x, e_x)}{H_t(e_x, e_x)}\right)^{1/2}.$$

If we write $\nabla_t + t\phi_t$ as the corresponding flat connection of $(\mathcal{E}, t\varphi)$ under the nonabelian Hodge correspondence, then

THEOREM 7.10 [Moc16].
On any compact subset K of $\Sigma \setminus Z$, $\mu^*_{C(x,t)} H_t$ converges smoothly to H^{Lim}. In addition, we have $\lim_{t\to 0} |(\nabla_t, \phi_t) - \Upsilon^{\mathrm{Moc}}(\mathcal{E},\varphi)|_{\mathcal{C}^k(K)} = 0$.

Comparing to the irreducible case Theorem 6.16, it is currently not known whether the convergence of (∇_t, ϕ_t) to $\Upsilon^{\mathrm{Moc}}(\mathcal{E},\varphi)$ is uniform.

Propositions 7.7 and 7.8 now give

THEOREM 7.11 (Theorem 1.3). *When $g \geq 3$, $\Upsilon^{\mathrm{Moc}}|_{\mathcal{M}_q^{\mathrm{st}}}$ is discontinuous, and when $g = 2$, $\Upsilon^{\mathrm{Moc}}|_{\mathcal{M}_q^{\mathrm{st}}}$ is continuous.*

8. The Compactified Kobayashi–Hitchin map

In this section, we define a compactified version of the Kobayashi–Hitchin map and prove the main theorem of our article. The Kobayashi–Hitchin map Ξ is a homeomorphism between the Dolbeault moduli space $\mathcal{M}_{\mathrm{Dol}}$ and the Hitchin moduli space $\mathcal{M}_{\mathrm{Hit}}$. We wish to extend this to a map $\overline{\Xi}$ from the compactified Dolbeault moduli space $\overline{\mathcal{M}}_{\mathrm{Dol}}$ to the compactification $\overline{\mathcal{M}}_{\mathrm{Hit}} \subset \mathcal{M}_{\mathrm{Hit}} \cup \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$ of the Hitchin moduli space and then to study the properties of this extended map.

8.1 The compactified Kobayashi–Hitchin map

We first summarise the results obtained above. By the construction in Section 4, there is an identification $\partial\overline{\mathcal{M}}_{\mathrm{Dol}} \cong (\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0))/\mathbb{C}^*$. Moreover, in (24)–(29), we have constructed the analytic Mochizuki map $\Upsilon^{\mathrm{Moc}} : \mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0) \to \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}$. Writing

$$(A^{\mathrm{Lim}}, \phi^{\mathrm{Lim}} = \varphi + \varphi^{\dagger_{\mathrm{Lim}}}) = \Upsilon^{\mathrm{Moc}}(\mathcal{E},\varphi)$$

for $w \in \mathbb{C}^*$ we then have

$$\Upsilon^{\mathrm{Moc}}(\mathcal{E}, w\varphi) = (A^{\mathrm{Lim}}, \phi^{\mathrm{Lim}} = w\varphi + \bar{w}\varphi^{\dagger_{\mathrm{Lim}}}).$$

Hence, Υ^{Moc} descends to a map $\partial\overline{\Xi}$ between $\mathbb{C}^*$ orbits:

$$\partial\overline{\Xi} : \partial\overline{\mathcal{M}}_{\mathrm{Dol}} = (\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0))/\mathbb{C}^* \longrightarrow \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*.$$

Together with the initial Kobayashi–Hitchin map $\Xi : \mathcal{M}_{\mathrm{Dol}} \to \mathcal{M}_{\mathrm{Hit}}$, we obtain (2):

$$\overline{\Xi} : \overline{\mathcal{M}}_{\mathrm{Dol}} = \mathcal{M}_{\mathrm{Dol}} \cup \partial\overline{\mathcal{M}}_{\mathrm{Dol}} \longrightarrow \mathcal{M}_{\mathrm{Hit}} \cup \mathcal{M}_{\mathrm{Hit}}^{\mathrm{Lim}}/\mathbb{C}^*. \tag{30}$$

Theorems 6.16 and 7.10 show that for a Higgs bundle $(\mathcal{E},\varphi) \in \mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0)$ and real t, $\lim_{t\to\infty} \Xi(\mathcal{E}, t\varphi) = \partial\overline{\Xi}[(\mathcal{E},\varphi)/\mathbb{C}^*]$. Thus the image of $\overline{\Xi}$ lies in $\overline{\mathcal{M}}_{\mathrm{Hit}}$, the closure of $\mathcal{M}_{\mathrm{Hit}}$ in $\mathcal{M}_{\mathrm{Hit}} \cup \mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0)$. There are natural extensions $\overline{\mathcal{H}}_{\mathrm{Dol}} : \overline{\mathcal{M}}_{\mathrm{Dol}} \to \overline{\mathcal{B}}$ and $\overline{\mathcal{H}}_{\mathrm{Hit}} : \overline{\mathcal{M}}_{\mathrm{Hit}} \to \overline{\mathcal{B}}$ such that $\overline{\mathcal{H}}_{\mathrm{Hit}} \circ \overline{\Xi} = \overline{\mathcal{H}}_{\mathrm{Dol}}$.

In summary, there are the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}} & \xrightarrow{\ \Xi\ } & \mathcal{M}_{\mathrm{Hit}} \\ \big\downarrow & & \big\downarrow \\ \overline{\mathcal{M}}_{\mathrm{Dol}} & \xrightarrow{\ \overline{\Xi}\ } & \overline{\mathcal{M}}_{\mathrm{Hit}} \end{array}, \qquad \begin{array}{ccc} \overline{\mathcal{M}}_{\mathrm{Dol}} & \xrightarrow{\ \overline{\Xi}\ } & \overline{\mathcal{M}}_{\mathrm{Hit}} \\ & \searrow^{\overline{\mathcal{H}}_{\mathrm{Dol}}} & \big\downarrow \overline{\mathcal{H}}_{\mathrm{Hit}} \\ & & \overline{\mathcal{B}} \end{array}.$$

We now turn to the analysis of some properties of the compactified Kobayashi–Hitchin map. Define

$$\overline{\mathcal{B}}^{\mathrm{reg}} = \{[(q,w)] \in \overline{\mathcal{B}} \mid q \neq 0 \,\text{has simple zeros}\}.$$

This is the compactified space of quadratic differentials with simple zeros. Let $\overline{\mathcal{B}}^{\mathrm{sing}} = \overline{\mathcal{B}} \setminus \overline{\mathcal{B}}^{\mathrm{reg}}$ be its complement. Additionally, define the open sets $\overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{reg}} = \overline{\mathcal{H}}_{\mathrm{Dol}}^{-1}(\overline{\mathcal{B}}^{\mathrm{reg}})$ and $\overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{reg}} = \overline{\mathcal{H}}_{\mathrm{Hit}}^{-1}(\overline{\mathcal{B}}^{\mathrm{reg}})$ as the collections of elements with regular spectral curves. Set $\overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{sing}} = \overline{\mathcal{H}}_{\mathrm{Dol}}^{-1}(\overline{\mathcal{B}}^{\mathrm{sing}})$ and $\overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{sing}} = \overline{\mathcal{H}}_{\mathrm{Hit}}^{-1}(\overline{\mathcal{B}}^{\mathrm{sing}})$ to be the sets of singular fibres. We can then write $\overline{\Xi} = \overline{\Xi}^{\mathrm{reg}} \cup \overline{\Xi}^{\mathrm{sing}}$, where

$$\overline{\Xi}^{\mathrm{reg}} : \overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{reg}} \longrightarrow \overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{reg}}, \quad \overline{\Xi}^{\mathrm{sing}} : \overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{sing}} \longrightarrow \overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{sing}}.$$

Proposition 8.1. *The map $\overline{\Xi}^{\mathrm{reg}} : \overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{reg}} \to \overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{reg}}$ is bijective, whereas $\overline{\Xi}^{\mathrm{sing}} : \overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{sing}} \to \overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{sing}}$ is neither surjective nor injective.*

Proof. The bijectivity of $\overline{\Xi}^{\mathrm{reg}}$ is established by Theorem 4.9. The non-surjectivity and non-injectivity of $\overline{\Xi}^{\mathrm{sing}}$ follow from Theorems 6.17 and 7.11. □

8.2 Discontinuity properties of the compactified Kobayashi–Hitchin map

In this subsection, we prove that the discontinuity of the compactified Kobayashi–Hitchin map (30) is fully determined by the discontinuity of the analytic Mochizuki map.

Let $(\mathcal{E}_i, t_i\varphi_i)$ be a sequence of Higgs bundles with real numbers $t_i \to +\infty$, $\det(\varphi_i) = q_i$, $Z_i = q_i^{-1}(0)$ and $\|q_i\|_{L^2} = 1$. We denote $\xi_i = [(\mathcal{E}_i, t_i\varphi_i)] \in \mathcal{M}_{\mathrm{Dol}}$. By the compactness of $\overline{\mathcal{M}}_{\mathrm{Dol}}$, after passing to a subsequence we may assume there is $\xi_\infty \in \partial\overline{\mathcal{M}}_{\mathrm{Dol}}$ such that $\lim_{i\to\infty} \xi_i = \xi_\infty$. Since $\partial\overline{\mathcal{M}}_{\mathrm{Dol}} \cong (\mathcal{M}_{\mathrm{Dol}} \setminus \mathcal{H}^{-1}(0))/\mathbb{C}^*$, we can select a representative $(\mathcal{E}_\infty, \varphi_\infty)$ of ξ_∞. By Proposition 7.9, we have that $(\mathcal{E}_i, \varphi_i)$ converges to $(\mathcal{E}_\infty, \varphi_\infty)$ in $\mathcal{M}_{\mathrm{Dol}}$ and that q_i converges to q_∞. We write $Z_\infty = q_\infty^{-1}(0)$. We note that for different choices of t_i, as long as $\lim_{i\to\infty} t_i = +\infty$ after passing to a subsequence, we always have $\lim_{i\to\infty} \xi = \xi_\infty \in \overline{\mathcal{M}}_{\mathrm{Dol}}$.

By Proposition 4.6, $\lim_{i\to\infty} \overline{\Xi}(\mathcal{E}_i, t_i\varphi_i)$ exists. The following result establishes the discontinuity of this map with respect to the analytic Mochizuki map Υ^{Moc}:

Proposition 8.2. *Under the previous conventions, suppose that q_i, q_∞ are irreducible. Consider $(\mathcal{E}_i, \varphi_i) \in \mathcal{M}_{q_i}$; if $\lim_{i\to\infty} \Upsilon^{\mathrm{Moc}}(\mathcal{E}_i, \varphi_i) \neq \Upsilon^{\mathrm{Moc}}(\mathcal{E}_\infty, \varphi_\infty)$, then there exist constants t_i such that for $\xi_i := (\mathcal{E}_i, t_i\varphi_i)$, we have $\lim_{i\to\infty} \overline{\Xi}(\xi_i) \neq \overline{\Xi}(\xi_\infty)$.*

Proof. Set

$$\begin{aligned} \overline{\Xi}(\mathcal{E}_i, t_i\varphi_i) &= \Xi(\mathcal{E}_i, t_i\varphi_i) = A_i + t_i\phi_i \\ \Upsilon^{\mathrm{Moc}}(\mathcal{E}_i, \varphi_i) &= (A_i^{\mathrm{Lim}}, \phi_i^{\mathrm{Lim}}) \; \textit{and} \\ \Upsilon^{\mathrm{Moc}}(\mathcal{E}_\infty, \varphi_\infty) &= (A_\infty^{\mathrm{Lim}}, \phi_\infty^{\mathrm{Lim}}), \end{aligned}$$

with t_i to be determined later and (A_i, ϕ_i) depending on t. Fixing a positive integer k, supposing $\lim_{i\to\infty} \Upsilon^{\mathrm{Moc}}(\mathcal{E}_i, \varphi_i) \neq \Upsilon^{\mathrm{Moc}}(\mathcal{E}_\infty, \varphi_\infty)$, and then passing to a subsequence, we could assume that there exist a compact set $K \subset \Sigma \setminus Z_\infty$ and $\epsilon_0 > 0$ such that $\|(A_i^{\mathrm{Lim}}, \phi_i^{\mathrm{Lim}}) - (A_\infty^{\mathrm{Lim}}, \phi_\infty^{\mathrm{Lim}})\|_{\mathcal{C}^k(K)} \geq \epsilon_0$ for $i \geq i_0'$.

By Theorem 6.16, for the fixed compact set K above and for each $(\mathcal{E}_i, \varphi_i)$, there exist t_i that are sufficiently large such that $\|(A_i, \phi_i) - (A_i^{\mathrm{Lim}}, \phi_i^{\mathrm{Lim}})\|_{\mathcal{C}^k(K)} < \frac{1}{4}\epsilon_0$. Moreover, by

Proposition 4.6, there is a limiting configuration $(A_\infty, \phi_\infty) := \lim_{i\to\infty}(A_i, \phi_i)$ defined over $\Sigma \setminus Z_\infty$ such that over K and for $i \geq i_0''$, we have

$$\|(A_i, \phi_i) - (A_\infty, \phi_\infty)\|_{\mathcal{C}^k(K)} < \frac{1}{4}\epsilon_0.$$

For $i \geq \max\{i_0', i_0''\}$, we compute

$$\begin{aligned} &\|(A_\infty, \phi_\infty) - (A_\infty^{\mathrm{Lim}}, \phi_\infty^{\mathrm{Lim}})\|_{\mathcal{C}^k(K)} \geq \|(A_i^{\mathrm{Lim}}, \phi_i^{\mathrm{Lim}}) - (A_\infty^{\mathrm{Lim}}, \phi_\infty^{\mathrm{Lim}})\|_{\mathcal{C}^k(K)} \\ &\quad - \|(A_\infty, \phi_\infty) - (A_i, \phi_i)\|_{\mathcal{C}^k(K)} - \|(A_i, \phi_i) - (A_i^{\mathrm{Lim}}, \phi_i^{\mathrm{Lim}})\|_{\mathcal{C}^k(K)} \\ &\geq \frac{1}{2}\epsilon_0. \end{aligned}$$

This proves the proposition. □

8.2.1 *Continuity along rays.* We now investigate the behavior of the compactified Kobayashi–Hitchin map when it is restricted to a singular fibre. Specifically, fix $0 \neq q \in H^0(K^2)$, and denote by $[q]$ the $\mathbb{C}^*$-orbit of $q \times 1$ in the compactified Hitchin base $\overline{\mathcal{B}}$. Define $\overline{\mathcal{M}}_{\mathrm{Dol},[q]} := \overline{\mathcal{H}}_{\mathrm{Dol}}^{-1}([q])$ and $\overline{\mathcal{M}}_{\mathrm{Hit},[q]} := \overline{\mathcal{H}}_{\mathrm{Hit}}^{-1}([q])$. Then the restriction of $\overline{\Xi}$ on $\overline{\mathcal{M}}_{\mathrm{Dol},[q]}$ defines a map $\overline{\Xi}_{[q]} : \overline{\mathcal{M}}_{\mathrm{Dol},[q]} \to \overline{\mathcal{M}}_{\mathrm{Hit},[q]}$.

THEOREM 8.3. *Let q be an irreducible quadratic differential.*

(i) The boundary map $\partial\overline{\Xi}_{[q]}|_{\overline{\mathcal{M}}_{\mathrm{Dol},[q]}}$ is continuous if q has only zeros of odd order and is discontinuous if q has at least one zero of even order.

(ii) If q has at least one zero of even order, then for each σ-divisor $D \neq 0$, there exists an even integer $n_D \geq 1$ so that for any Higgs bundle $(\mathcal{F}, \psi) \in \mathcal{M}_{q,D}$, there exist $2n_D$ sequences of Higgs bundles $(\mathcal{E}_i^k, \varphi_i^k)$, $k = 1, \ldots, 2n_D$ such that

* $\lim_{i\to\infty}(\mathcal{E}_i^k, \varphi_i^k) = (\mathcal{F}, \psi)$ *for* $k = 1, \ldots, 2n_D$;
* *and if we write*

$$\eta^k := \lim_{i\to\infty} \partial\overline{\Xi}_{[q]}(\mathcal{E}_i^k, t_i\varphi_i^k) \quad \text{, then} \quad \xi := \lim_{i\to\infty} \partial\overline{\Xi}_{[q]}(\mathcal{F}, t_i\psi)$$

  - *if $(\mathcal{F}, \psi)$ doesn't lie in the real locus, then $\xi, \eta^1, \ldots, \eta^{2n_D}$ are $2n_D + 1$ different limiting configurations,*
  - *if $(\mathcal{F}, \psi)$ lies in the real locus, then $\eta^i \cong \eta^{n_D+i}$ for $i = 1, \cdots, n$ and we obtain $n_D + 1$ different limiting configurations.*

* *for each k, there exists constants $t_i \to +\infty$ such that $\lim_{i\to\infty} \overline{\Xi}_{[q]}(\mathcal{E}_i^k, t_i\varphi_i^k) \neq \overline{\Xi}_{[q]}(\mathcal{F}, \psi)$.*

Proof. This follows from Theorem 6.17, Proposition 6.15 and Proposition 8.2. □

8.2.2 *Varying fibre.* With the conventions above, suppose that $(\mathcal{E}_i, \varphi_i)$ converges to $(\mathcal{E}_\infty, \varphi_\infty)$, with q_∞ having only simple zeros, and that $\xi_i = (\mathcal{E}_i, t_i\varphi_i)$ converges to ξ_∞ on $\overline{\mathcal{M}}_{\mathrm{Dol}}$. Since the condition of having only simple zeros is open, the q_i's also have simple zeros when i is sufficiently large.

Proposition 8.4. *Suppose q_∞ has only simple zeros. Then $\lim_{i\to\infty}\overline{\Xi}(\xi_i)=\overline{\Xi}(\xi_\infty)$. In particular, the map $\overline{\Xi}^{\mathrm{reg}}:\overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{reg}}\to\overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{reg}}$ is continuous.*

Proof. Let S_i denote the spectral curve of $(\mathcal{E}_i,\varphi_i)$, with branching locus Z_i. Also, let $L_i:=\chi_{\mathrm{BNR}}^{-1}(\mathcal{E}_i,\varphi_i)$ be the eigenline bundles. By the construction in Section 6, we have $\Upsilon^{\mathrm{Moc}}(\xi_i)=\mathcal{F}_*(L_i,\chi_i)$, where $\chi_i=-\frac{1}{2}\chi_{Z_i}$. Our assumption implies that $\mathcal{F}_*(L_i,\chi_i)$ converges to $\mathcal{F}_*(L_\infty,\chi_\infty)$ in the sense of Definition 3.2. Thus, by Theorem 3.3, we obtain the convergence of the limiting configurations: $\lim_{i\to\infty}\Upsilon^{\mathrm{Moc}}(\xi_i)=\Upsilon^{\mathrm{Moc}}(\xi_\infty)$. The claim follows from Proposition 8.2. □

Theorem 8.5. *The map $\overline{\Xi}^{\mathrm{reg}}:\overline{\mathcal{M}}_{\mathrm{Dol}}^{\mathrm{reg}}\to\overline{\mathcal{M}}_{\mathrm{Hit}}^{\mathrm{reg}}$ is a homeomorphism.*

Proof. By Theorem 4.9, $\overline{\Xi}^{\mathrm{reg}}$ is a bijection. Moreover, by Proposition 8.4, $\overline{\Xi}^{\mathrm{reg}}$ is continuous. Finally, that $(\overline{\Xi}^{\mathrm{reg}})^{-1}$ is continuous follows directly from the construction in [MSWW19]. □

Appendix A. Classification of rank 1 torsion-free modules for A_n singularities

In this appendix, we review the classification result for rank 1 torsion-free modules at A_n singularities, as given in [GK85]. We compute the integer invariants defined in Section 5.3.

Let S be the spectral curve of an $\mathrm{SL}(2,\mathbb{C})$ Higgs bundle, and let x be a singular point with local defining equation given by $r^2-s^{n+1}=0$; this is an A_n singularity. Let $p:\widetilde{S}\to S$ be the normalisation, where $p^{-1}(x)=\{\tilde{x}_+,\tilde{x}_-\}$ if n is odd and $p^{-1}(x)=\tilde{x}$ if n is even. We use R to denote the completion of the local ring $\mathcal{O}_x$, K to denote its field of fractions and $\widetilde{R}$ to denote its normalization.

A.1 A_{2n} singularity

The local equation is $r^2-s^{2n+1}=0$. The normalization induces a map between coordinate rings, and we can write

$$\psi:\mathbb{C}[r,s]/(r^2-s^{2n+1})\longrightarrow\mathbb{C}[t],\quad \psi(f(r,s))=f(t^{2n+1},t^2),$$

where $\widetilde{R}=\mathbb{C}[[t]]$ and $R=\mathbb{C}[[t^2,t^{2n+1}]]\subset\widetilde{R}$. According to [GK85, Anh. (1.1)], any rank 1 torsion-free R-module can be written as

$$M_k=R+R\cdot t^k\subset\widetilde{R},\quad k=1,3,\dots,2n+1.$$

Here, M_k is a fractional ideal that satisfies $R\subset M_k\subset\widetilde{R}$, with $M_1=\widetilde{R}$ and $M_{2n+1}=R$. We may express any $f\in M_k$ as $f=\sum_{i=0}^{\frac{k-1}{2}}f_{2i}t^{2i}+\sum_{i\geq k}f_it^i$, where $f_i\in\mathbb{C}$.

We are interested in the integers $\ell_x:=\dim_{\mathbb{C}}(M_k/R)$, $a_{\tilde{x}}:=\dim_{\mathbb{C}}(\widetilde{R}/C(M_k))$ and $b_x=\dim_{\mathbb{C}}(T(M_k\otimes_R\widetilde{R}))$ (where T denotes the torsion-free submodule). Thus, as a $\mathbb{C}$-vector space, M_k/R is generated by $t^k,t^{k+2},\dots,t^{2n-1}$, implying that $\ell_x=\frac{2n+1-k}{2}$.

The conductor of M_k is given by $C(M_k)=\{u\in K\mid u\cdot\widetilde{R}\subset M_k\}$. By the expression of M_k and a straightforward computation, we have $C(M_k)=(t^{k-1})$, where (t^{k-1}) is the ideal in $\widetilde{R}$ generated by t^{k-1}. Thus, $1,t,\dots,t^{k-2}$ will form a basis for $\widetilde{R}/C(M_k)$, and we have $a_{\tilde{x}}=k-1$. Therefore, we have $a_{\tilde{x}}=2n-2\ell_x$.

For $i=0,1,\dots,\frac{2n-1-k}{2}$, we define $s_i=t^{k+2i}\otimes_R 1-1\otimes t^{k+2i}\in M_k\otimes_R\widetilde{R}$. As k is odd, $t^{2n+1-k-2i}\in R$ and $t^{2n+1-k-2i}s_i=t^{2n+1}\otimes_R 1-1\otimes_R t^{2n+1}=0$, where the last equality occurs because $t^{2n+1}\in R$. Moreover, $\{s_1,\dots,s_{\frac{2n-1-k}{2}}\}$ form a basis of $T(M_k\otimes_R\widetilde{R})$; thus $b_x=\frac{2n+1-k}{2}=\ell_x$.

A.2 A_{2n-1} singularity

The local equation is $r^2 - s^{2n} = 0$. The normalisation induces a map between the coordinate rings:

$$\psi : \mathbb{C}[r,s]/(r^2 - s^{2n}) \longrightarrow \mathbb{C}[t] \oplus \mathbb{C}[t], \quad \psi(f(r,s)) = (f(t^n, t), f(-t^n, t))$$

where $\widetilde{R} = \mathbb{C}[[t]] \oplus \mathbb{C}[[t]]$ and $R = \mathbb{C}[[(t,t),(t^n,-t^n)]] \cong \mathbb{C}[[(t,t),(t^n,0)]]$. By [GK85, Anh. (2.1)], any rank 1 torsion-free R-module can be written as:

$$M_k = R + R \cdot (t^k, 0) \subset \widetilde{R}, \quad k = 0, 1, \ldots, n.$$

Then M_k is also a fractional ideal, with $R \subset M_k \subset \widetilde{R}$. Moreover, $M_n = R$, and $M_0 = \widetilde{R}$.

As $p^{-1}(x) = \{\tilde{x}_+, \tilde{x}_-\}$, $\widetilde{R}$ contains two maximal ideals, $\mathfrak{m}_+ = ((t,1))$, $\mathfrak{m}_- = ((1,t))$. For $f \in M_k$, we can express f as:

$$f = \sum_{i=0}^{k-1} f_{ii}(t^i, t^i) + \sum_{l \geq 0} f_{l0}(t^{k+l}, 0) + f_{0l}(0, t^{k+l}),$$

where $f_{ij} \in \mathbb{C}$. Therefore, $\ell_x = \dim_{\mathbb{C}}(M_k/R) = n - k$. Moreover, using this expression, we can compute the conductor $C(M_k) = ((t^k,1)) \cdot ((1,t^k))$, which implies $a_{\tilde{x}_\pm} = k$. Similarly, for $i = k, \ldots, n-1$, we define $s_i = (t^i, 0) \otimes_R (1,1) - (1,1) \otimes_R (t^i, 0)$; then $(t,t)^{n-i} \cdot s_i = 0$, and $\{s_k, \ldots, s_{n-1}\}$ will be a basis for $T(M_k \otimes_R \widetilde{R})$ and $b_x = \ell_x$.

In summary, we have the following:

Proposition 9.2.1. *For the integers defined above, we have:*

(i) for the A_{2n} singularity, we have $a_{\tilde{x}} = 2n - 2\ell_x$ and $b_x = \ell_x$;
(ii) for the A_{2n-1} singularity, we have $a_{\tilde{x}_\pm} = n - \ell_x$ and $b_x = \ell_x$.

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Conflicts of Interest

None.

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